Planar Lorentz gas walk in a random scenery

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Abstract. *We consider the planar Lorentz gas with finite horizon. The random scenery is given by a sequence of independent, identically distributed, centered and square integrable random variables. To each obstacle, we associate one of these random variables. We suppose that each time the particule hits an obstacle, it wins the amount given by the random variable associated to the obstacle. We prove a convergence in distribution to a Wiener process for the amount won by the particle (normalized by $\sqrt{t\log(t)}$) when the time goes to infinity. The convergence is established for any probability measure absolutely continuous with respect to the usual σ-finite invariant measure. Such a result has been established by Bolthausen [3] in the case of the simple random walk in $\mathbb{Z}^2$. We follow the scheme of his proof. We compensate the lack of independence by two extensions of the local limit theorem proved by Szasz and Varjù in [25]. This paper answers a question of Szàsz about the asymptotic behaviour of $\sum_{k=0}^{n-1}\zeta_k$ where $(\zeta_\ell)_\ell$ is a sequence of iid centered square integrable random variables and where S_k is the number of the cell at the kth reflection.*

1 Introduction

Since the early work of Sinai [24], the billiard system considered here has been studied by many authors ([4, 5, 6, 7, 12] and others). Let us introduce this dynamical system in infinite measure.

In $\mathbb{R}^2$, we consider a finite number of convex open sets $O_1, ..., O_I$, with boundary C^3-smooth and with non null curvature. We repeat these sets $\mathbb{Z}^2$-periodically by considering $U_{i,\ell} = j + O_i$ for all $i \in \{1, ..., I\}$ and all $\ell \in \mathbb{Z}^2$. We suppose that the closures of the $U_{i,\ell}$ are pairwise disjoint. For any $\ell \in \mathbb{Z}^2$, we call ℓ-cell the union $\bigcup_{i=1}^I \partial U_{i,\ell}$. The random scenery is given by a sequence of independent identically distributed real-valued, centered and square integrable random variables $(\zeta_{(i,\ell)})_{i\in\{1,...,I\},\,\ell\in\mathbb{Z}^2}$. We associate the value $\zeta_{(i,\ell)}$ to the obstacle $U_{i,\ell}$. Let us consider a point particle moving in the domain $Q := \mathbb{R}^2 \setminus \bigcup_{i=1}^I \bigcup_{\ell\in\mathbb{Z}^2} U_{i,\ell}$ with unit speed and with elastic reflections off ∂Q. We associate to the particle an amount equal to 0 at time 0. This amount only changes at reflection times : the particle wins $\zeta_{(i,\ell)}$ each time it hits $U_{i,\ell}$. We are interested in the asymptotic behaviour of the amount when "the time" goes to infinity. We will envisage "the time" in two ways : continuous time and discrete time. We will define $\tilde{Z}_t$ as the total amount of the particle at time t and Z_n the total amount of the particle at the n^{th} reflection time.

1.1 Billiard flow $(\mathcal{M}_1, \mu_1, (Y_t)_t)$ and billiard transformation (M, ν, T) in the plane

We call configuration of a particle at some time its position-speed couple. When a reflection occurs, there is coexistence of two configurations : one corresponding to the incident vector and one corresponding to the reflected vector. To avoid ambiguity, we will only consider reflected vectors. Hence the set of configurations (position-speed couples) will be :

$$\mathcal{M}_1 := \{(q, \vec{v}) \in Q \times \mathbb{R}^2 \ : \ \|\vec{v}\| = 1; \ \ q \in \partial Q \ \Rightarrow \ \langle \vec{n}(q), \vec{v} \rangle \geq 0\},$$

with $\vec{n}(q)$ the unit vector normal to ∂Q at $q \in \partial Q$ oriented to the inside of Q. The billiard flow $(Y_t)_t$ is the flow on $\mathcal{M}_1$ such that $Y_t(q, \vec{v}) = (q_t, \vec{v}_t)$ is the configuration at time t of a particle with configuration $(q, \vec{v})$ at time 0. The billiard flow preserves the Lebesgue measure μ_1 on $\mathcal{M}_1$. Now we only consider reflection times. Let M be the set of reflected vectors off ∂Q :

$$M := \{(q, \vec{v}) \in \partial Q \times \mathbb{R}^2 \ : \ \|\vec{v}\| = 1 \text{ and } \langle \vec{n}(q), \vec{v} \rangle \geq 0\}.$$

A point $(q, \vec{v}) \in M$ is parametrized by (i, r, φ, ℓ) if $q - \ell$ is the point of O_i with curvilinear absciss r and if φ is the angular measure of $(\vec{n}(q), \vec{v})$ taken in $[-\frac{\pi}{2}; \frac{\pi}{2}]$. The billiard transformation T maps a configuration $y \in M$ at a reflection time to the configuration $T(y) = y'$ corresponding to the next reflection off ∂Q. This transformation preserves the measure ν given by $d\nu(q, \vec{v}) = \cos(\varphi)\, dr d\varphi$, with the parametrization (i, r, φ, ℓ) of $(q, \vec{v}) \in M$.

We define the function $\tau : M \to [0; +\infty[$ by : $\tau(q, \vec{v}) := \min\{s > 0 \ : \ q + s\vec{v} \in \partial Q\}$. The quantity $\tau(q, \vec{v})$ corresponds to the time before the next reflection off ∂Q. **Here, we suppose that the billiard system has finite horizon**, i.e. $\sup \tau < +\infty$. We already know that this system is recurrent (see the works of Conze in [10], of Schmidt in [21] and of Szàsz and Varjù in [25]) and that it is totally ergodic (see [23] and [19]). Others have been got by Dolgopyat, Szàsz and Varjù in [11]. The billiard flow $(\mathcal{M}_1, \mu_1, (Y_t)_t)$ can be represented as the special flow over (M, ν, T) with roof function τ. Let us explicit this. Let us define $\tilde{\mathcal{M}}_1 := \{(y, s) \ : \ y \in M; \ 0 \leq s < \tau(y)\}$ endowed with the measure $\tilde{\mu}_1$ given by : $d\tilde{\mu}_1(y, s) = d\nu(y) ds$. Let $(\tilde{Y}_t)_t$ be the flow defined on $\tilde{M}_1$ by $\tilde{Y}_t(y, s) = (y, s + t)$ with the identifications $(y, \tau(y)) \equiv (T(y), 0)$. Let $\Delta : \tilde{\mathcal{M}}_1 \to \mathcal{M}_1$ given by : $\Delta((q, \vec{v}), s) = (q + s\vec{v}, \vec{v})$. This bi-measurable function satisfies : $Y_t = \Delta \circ \tilde{Y}_t \circ \Delta^{-1}$ and $\Delta_*(\tilde{\mu}_1) = \mu_1$.

1.2 Billiard transformation in the torus $(\bar{M}, \bar{\nu}, \bar{T})$

Let us define $\bar{M} = \{(q, \vec{v}) \in M \ : \ q \in \bigcup_{i=1}^{I} \partial O_i\}$ and $\bar{T} : \bar{M} \to \bar{M}$ with $\bar{T}(q, \vec{v}) = (q', \vec{v}')$ if there exists $\ell \in \mathbb{Z}^2$ such that $T(q, \vec{v}) = (q' + \ell, \vec{v})$. Let $\bar{\nu}$ be the probability measure on $\bar{M}$ proportional to the restriction of ν to $\bar{M}$.

The study of this system is complicated by the discontinuities of the transformation $\bar{T}$. But it is known that $\bar{T}$ is C^1-regular on $\bar{M} \setminus (R_0 \cup \bar{T}^{-1}(R_0))$, where the set $R_0 := \{(q, \vec{v}) \in \bar{M} \ : \ \langle \vec{v}, \vec{n}(q) \rangle = 0\}$ is the set of tangent vectors.

1.3 Lorentz gas walk in a random scenery

It is easy to see that the billiard system (M, ν, T) is a cylindrical extension of the billiard system $(\bar{M}, \bar{\nu}, \bar{T})$ by some function $\Phi : \bar{M} \to \mathbb{Z}^2$. For any $(q, \vec{v}) \in \bar{M}$ and any $\ell \in \mathbb{Z}^2$, we have $T(q + \ell, \vec{v}) =$

$(q'+\ell+\Phi(q,\vec{v}),\vec{v}')$ with $(q',\vec{v}')=\bar{T}(q,\vec{v})$ and $T^n(q+\ell,\vec{v})=\left(q_n+\ell+\sum_{k=0}^{n-1}\Phi(\bar{T}^k(q,\vec{v})),\vec{v}_n\right)$ with $(q_n,\vec{v}_n)=\bar{T}^n(q,\vec{v})$.

In the sequel, identify M with $\bar{M}\times\mathbb{Z}^2$ by the one-to-one map $\Pi_0:\bar{M}\times\mathbb{Z}^2\to M$ given by : $\Pi_0((q,\vec{v}),\ell)=(q+\ell,\vec{v})$. We notice that the image measure of ν by Π_0^{-1} is $\bar{\nu}\otimes\sum_{\ell\in\mathbb{Z}^2}\delta_\ell$ (where δ_ℓ is the Dirac measure in ℓ).

Let us consider the asymptotic covariance matrix Σ^2 associated to Φ :

$$\Sigma^2:=\lim_{n\to+\infty}Cov\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\Phi\circ\bar{T}^k\right).$$

Because of the recurrence of the Lorentz gas, the matrix Σ^2 is invertible. Let us write :

$$S_0:=0\ \text{ and }\ S_n:=\sum_{k=0}^{n-1}\Phi\circ\bar{T}^k.$$

We will also define the random variable $\mathcal{I}_k$ equal to the number of the obstacle (taken in $1,...,I$) met by the particle at the k^{th} reflection off an obstacle. Let us consider a sequence of independent identically distributed random variables $(\zeta_{i,\ell})_{i=1,...,I,\ \ell\in\mathbb{Z}^2}$ defined on some probability space $(\Omega_0,\mathcal{F}_0,\mathbb{P}_0)$. We suppose that $\zeta_{i,\ell}$ have zero mean and are square integrable with variance $\sigma^2>0$.

We define the sequence of random variables $(Z_n)_n$ on the direct product $(\Omega,\mathcal{F}):=(\bar{M}\times\Omega_0,\mathcal{B}(\bar{M}\times\mathbb{Z}^2)\otimes\mathcal{F}_0)$ by :

$$\forall(x,\ell_0)\in\bar{M}\times\mathbb{Z}^2,\ \forall\omega\in\Omega_0,\quad Z_n(\Pi_0(x,\ell_0),\omega):=\sum_{k=1}^{n}\zeta_{\mathcal{I}_k(x),\ell_0+S_k(x)}(\omega).$$

We are interested in the asymptotic behaviour of Z_n when n goes to infinity. We establish a result of convergence in distribution for any probability measure $(h\nu)\otimes\mathbb{P}_0$ on $(\Omega,\mathcal{F})$.

Theorem 1 *Let $h:M\to\mathbb{R}$ be any positive ν-integrable function such that $\int_M h\,d\nu=1$. The sequence of processes* $\left(\left(\sqrt{\frac{2\pi\sqrt{det(\Sigma^2)}(\sum_{i=1}^{I}length(\partial O_i))^2}{\sum_{i=1}^{I}(length(\partial O_i))^2 n\log(n)\sigma^2}}Z_{\lfloor nt\rfloor}\right)_{t\geq 0}\right)_{n\geq 1}$ *converges weakly (in $D([0,\infty))$) to the Wiener process (for the probabilities measures $h\nu\otimes\mathbb{P}_0$.*

With the same proof, we can answer to the question of Szász by proving that the sequence of processes $\left(\left(\sqrt{\frac{2\pi\sqrt{det(\Sigma^2)}}{n\log(n)\sigma^2}}\sum_{k=1}^{\lfloor nt\rfloor}\zeta_{1,S_k}\right)_{t\geq 0}\right)_{n\geq 1}$ converges weakly (in $D([0,\infty))$) to the Wiener process (for the same probabilities measures $h\nu\otimes\mathbb{P}_0$).

We define the sequence of random variables $(\tilde{Z}_t)_{t\geq 0}$ on the direct product $(\Omega',\mathcal{F}'):=(\mathcal{M}_1\times\Omega_0,\mathcal{B}(\mathcal{M}_1)\otimes\mathcal{F}_0)$ by :

$$\forall(q,\vec{v})\in M,\ \forall s\in[0,\tau(q,\vec{v})),\quad \tilde{Z}_t\left((q+s\vec{v},\vec{v}),\omega\right)=Z_{\tilde{n}(t+s,(q,\vec{v}))}((q,\vec{v}),\omega),$$

with

$$\tilde{n}(u,(q,\vec{v})):=\sup_{n\geq 0}\left\{n\geq 0\ :\ \sum_{k=0}^{n-1}\tau(q,\vec{v})\leq t\right\}$$

(representing the number of reflections before time u for a particle starting with configuration $(q,\vec{v})$ at time 0). We establish a result of convergence in distribution for $\tilde{Z}_t$ for any probability measure $(g\mu_1)\otimes\mathbb{P}_0$ on $(\Omega',\mathcal{F}')$.

Corollary 2 *Let $g:\mathcal{M}_1\to\mathbb{R}$ be any positive μ_1-integrable function such that $\int_{\mathcal{M}_1} g\,d\mu=1$. The sequence of processes* $\left(\left(\sqrt{\frac{2\pi\sqrt{det(\Sigma^2)}(\sum_{i=1}^I length(\partial O_i))^2\int_{\bar{M}}\tau\,d\bar{\nu}}{\sum_{i=1}^I(length(\partial O_i))^2 t\log(t)\sigma^2}}\tilde{Z}_{nt}\right)_{t\ge 0}\right)_{n\ge 1}$ *converges weakly (in $D[0,\infty)$) to the Wiener process (for the probability measure $g\mu_1\otimes\mathbb{P}_0$ on $\mathcal{M}_1\times\Omega_0$).*

2 Proof of our results

2.1 Tools

As we said briefly in the abstract, we will use the scheme of the proof of Bolthausen [3]. We will compensate the lack of independence by two refinements of the local limit theorem established by Szász and Varjú in [25] (see the appendix for the proofs).

Proposition 3 *There exists a real number $C>0$ such that, for all nonnegative integers n, m and k and for any $i,j,i',j'\in\{1,\dots,I\}$ and for any $N_1,N_2\in\mathbf{Z}^2$ we have :*

$$\left|Cov_{\bar{\nu}}(\mathbf{1}_{\{\mathcal{I}_0=i,\ S_n=N_1,\ \mathcal{I}_n=i'\}},\mathbf{1}_{\{\mathcal{I}_{n+m}=j,\ S_{n+m+k}-S_{n+m}=N_2,\ \mathcal{I}_{n+m+k}=j'\}}\right|\le\frac{C\tau_1^m}{(n+1)(k+1)}.$$

Proposition 4 *Let any real number $p>1$. There exist $C>0$ and $K_0>0$ such that, for any positive integer k, any measurable set B such that, if $x\in B$ then $\gamma^s(x)\subseteq B$, for any positive integer r and any measurable set A union of connected components of $\bar{M}\setminus\bigcup_{i=1}^r\bar{T}^{-i}(R_0)$, then for any $N\in\mathbb{Z}^2$, we have :*

$$\left|\bar{\nu}\left(A\cap\bar{T}^{-(k+r)}(B)\cap\{S_{k+r}-S_r=N\}\right)-\frac{\bar{\nu}(A)\bar{\nu}(B)}{\sqrt{det(\Sigma^2)}2\pi k}e^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}N,N\rangle}\right|\le$$
$$\le K_0\left(\frac{\bar{\nu}(B)+\bar{\nu}(A)\bar{\nu}(B)^{1/p}}{k^{3/2}}\left(\frac{\|N\|}{\sqrt{k}}+\frac{\|N\|^3}{k^{3/2}}\right)\exp^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}N,N\rangle}+\frac{\bar{\nu}(B)^{1/p}}{k^2}\right). \tag{1}$$

The fact that, by our method, we cannot take $p=1$ will complicate our calculations. In the independent case as in more friendly cases (such as subshifts of finite type), the estimations are got with $p=1$. The condition $p>1$ comes from the fact that the Young norm does not dominate $\|\cdot\|_\infty$ but can be chosen so that it dominates $\|\cdot\|_q$ (for any arbitrary real number $q>1$). So the fact that we take $p\neq 1$ is due to the method used here and might certainly be improved by another approach.

2.2 Scheme of the proof of theorem 1

Let us notice that, for every $(x,\ell_0)\in\bar{M}\times\mathbb{Z}^2$, with respect to $\mathbb{P}_0$, $Z_n(\Pi_0(x,\ell_0),\cdot)$ has zero mean and its (conditional) variance is $\sigma^2V_n(x)$ with $V_n:=\sum_{k,\ell=1}^n\mathbf{1}_{\{S_k=S_\ell\text{ and }\mathcal{I}_k=\mathcal{I}_\ell\}}$. Hence the study of V_n will be useful.

Proposition 5 *We have :*

$$\mathbb{E}_{\bar\nu}[V_n] \sim_{n\to+\infty} \frac{\sum_{i=1}^I (length(\partial O_i))^2}{(\sum_{i=1}^I length(\partial O_i))^2}\frac{n\log(n)}{2\pi\sqrt{det(\Sigma^2)}}.$$

Proof. We have : $\mathbb{E}_{\bar\nu}[V_n] = n + 2\sum_{i=1}^I\sum_{k=1}^{n-1}(n-k)\bar\nu(\mathcal{I}_0 = i,\ S_k = 0,\ \mathcal{I}_k = i)$. But we know that $\bar\nu(\mathcal{I}_0 = i,\ S_k = 0,\ \mathcal{I}_k = i)$ is equivalent to $\frac{(\bar\nu(\mathcal{I}_0=i))^2}{2\pi\sqrt{det(\Sigma^2)}k}$ when k goes to infinity (according to proposition 4), *qed.*

The more technical point of our proof will be the proof of :

Proposition 6 *We have : $Var_{\bar\nu}(V_n) = O(n^2\log(n))$.*

For all $(x,\ell_0)\in\bar M\times\mathbb{Z}^2$ and all $\omega\in\Omega_0$,we have :

$$Z_n(\Pi_0(x,\ell_0),\omega) = \sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\zeta_{i,\ell+\ell_0}(\omega)\mathcal{N}_{i,\ell}(n)(x),$$

with $\mathcal{N}_{i,\ell}(n) := \sum_{k=1}^n \mathbf{1}_{\{S_k=\ell \text{ and } \mathcal{I}_k=i\}}$ the number of reflection off $U_{i,\ell}$ before time n. For all $\ell\in\mathbb{Z}^2$, let us define :

$$\mathcal{N}_\ell(n) := \sum_{k=1}^n \mathbf{1}_{\{S_k=\ell\}}.$$

We clearly have : $\mathcal{N}_\ell(n) = \sum_{i=1,\dots,I}\mathcal{N}_{i,\ell}$. This will be useful in our calculations. To simplify notations, for any integer $n\ge1$, we write :

$$d_n := \sqrt{\frac{\sum_{i=1}^I(\mathrm{length}(\partial O_i))^2 n\log(n)\sigma^2}{2\pi\sqrt{det(\Sigma^2)}(\sum_{i=1}^I\mathrm{length}(\partial O_i))^2}}.$$

1. **Convergence of the finite dimensional distributions.**

   Let $m\ge1$, $a_1,\dots,a_m\in\mathbb{R}$ and $0=t_0<t_1<\dots<t_m$. We have :

   $$\sum_{j=1}^m a_j\left(Z_{\lfloor nt_j\rfloor} - Z_{\lfloor nt_{j-1}\rfloor}\right) = \sum_{j=1}^m\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2} a_j\left(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor) - \mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor)\right)\zeta_{i,\ell+\ell_0}.$$

   Following [3], we will apply the Lindeberg theorem (see section 27 of [1]).

   For (x,ℓ_0) fixed in $\bar M\backslash\mathbb{Z}^2$, this random variable taken in $(\Pi_0(x,\ell),\cdot)$ is a sum of independent (but not identically distributed) random variables (with respect to $\mathcal{P}_0$). We will prove that we have :

   Proposition 7 *ν-almost everywhere, for any $a>0$, $\sup_{i=1,\dots,I;\ \ell\in\mathbb{Z}^2}\mathcal{N}_{i,\ell}(n) = o(n^a)$.*

   This allows us to apply the Lindeberg theorem. Hence, for $\bar\nu$-almost every $x\in\bar M$, for all $\ell_0\in\mathbb{Z}^2$, we get that, the random variable :

   $$\hat Z_n = \frac{\sum_{j=1}^m a_j(Z_{\lfloor nt_j\rfloor}(\Pi_0(x,\ell_0),\cdot) - Z_{\lfloor nt_{j-1}\rfloor}(\Pi_0(x,\ell_0),\cdot))}{\sqrt{\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\left(\sum_{j=1}^m a_j\left(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor)(x) - \mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor)(x)\right)\right)^2\sigma^2}}$$

converges in distribution (with respect to $\mathbb{P}_0$) to a gaussian centered random variable with variance 1 (as n goes to infinity). Hence, $\hat{Z}_n$ converges in distribution (with respect to $(h\nu)\otimes\mathbb{P}_0$) to gaussian centered random variable with variance 1 (as n goes to infinity). We will prove that :

Proposition 8 *The sequence of random variables*

$$\left(\frac{(d_n)^2}{\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\left(\sum_{j=1}^m a_j\left(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor)-\mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor)\right)\right)^2\sigma^2}\right)_{n\geq 1}$$

converges in probability (for $h\nu$) to $\sum_{j=1}^m (a_j)^2(t_j-t_{j-1})$ as n goes to infinity.

Proof. Since the random variables considered here only depend on $x\in\bar{M}$ and not on the number ℓ_0 of the cell, it suffices to prove the result for the measure $\bar{\nu}$. Let us notice that if $m=1$ then this means that $\frac{(d_n)^2}{V_n}$ converges in probability to 1, which is an immediate consequence of propositions 5 and 6. But it is more complicated when $m\geq 2$. Let us notice that :

$$\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\left(\sum_{j=1}^m a_j\left(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor)-\mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor)\right)\right)^2=\Delta+\Gamma,$$

with $\Delta:=\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\sum_{j=1}^m(a_i)^2(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor)-\mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor))^2$ and

$$\begin{aligned}\Gamma &:= 2\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\sum_{1\leq j<j'\leq m}a_ja_{j'}\sum_{k=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j\rfloor}\sum_{k'=\lfloor nt_{j'-1}\rfloor+1}^{\lfloor nt_{j'}\rfloor}\mathbf{1}_{\{S_k=\ell,\ \mathcal{I}_k=i,\ S_{k'}=\ell,\ \mathcal{I}_{k'}=i\}}\\ &= 2\sum_{1\leq j<j'\leq m}a_ja_{j'}\sum_{k=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j\rfloor}\sum_{k'=\lfloor nt_{j'-1}\rfloor+1}^{\lfloor nt_{j'}\rfloor}\mathbf{1}_{\{S_k=S_{k'},\ \mathcal{I}_k=\mathcal{I}_{k'}\}}.\end{aligned}$$

- First, let us notice that we have : $\mathbb{E}[\Delta]=\sum_{j=1}^m(a_j)^2\mathbb{E}\left[V_{\lfloor nt_j\rfloor-\lfloor nt_{j-1}\rfloor}\right]$ and :

$$\begin{aligned}\mathbb{E}[\Gamma] &\leq 2\sum_{1\leq j<j'\leq m}|a_ja_{j'}|\sum_{k=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j\rfloor}\sum_{k'=\lfloor nt_{j'-1}\rfloor+1}^{\lfloor nt_{j'}\rfloor}\mathbb{E}\left[\mathbf{1}_{\{S_{k'}=S_k,\ \mathcal{I}_{k'}=\mathcal{I}_k\}}\right]\\ &\leq 2\sup(|a_1|,...,|a_m|)^2\sum_{j=1}^m\sum_{\ell=1}^{\lfloor nt_m\rfloor-\lfloor nt_{j-1}\rfloor}\ell\mathbb{P}(S_\ell=0)\leq O(n),\end{aligned}$$

since $\mathbb{P}(S_\ell=0)=O\left(\frac{1}{\ell}\right)$. Hence, we have proven that :

$$\mathbb{E}\left[\sum_{i=1}^I\sum_{\ell\in\mathbb{Z}^2}\left(\sum_{j=1}^m a_j\left(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor)-\mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor)\right)\right)^2\right]\sigma^2$$

is equivalent to ${d_n}^2\sum_{j=1}^m(a_j)^2(t_j-t_{j-1})$.

- Now, let us prove that $Var\left(\sum_{i=1}^{I}\sum_{\ell\in\mathbb{Z}^2}\left(\sum_{j=1}^{m} a_j\left(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor)-\mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor)\right)\right)^2\right)$ is in $o((d_n)^4)$. We have : $Var(\Delta+\Gamma)\le 2Var(\Delta)+Var(\Gamma)$. Let us start by bounding $Var(\Delta)$. Let us notice that we have $\Delta=\sum_{j=1}^{m}\Delta_j$ with :
$$\Delta_j=\sum_{i=1}^{I}\sum_{\ell\in\mathbb{Z}^2}(a_i)^2(\mathcal{N}_{i,\ell}(\lfloor nt_j\rfloor)-\mathcal{N}_{i,\ell}(\lfloor nt_{j-1}\rfloor))^2.$$
Let $j=1,...,m$. We have : $Var(\Delta_j)=Var(V_{\lfloor nt_j\rfloor-\lfloor nt_{j-1}\rfloor})$. Hence, according to proposition 6, $Var(\Delta)$ is in $O(n^2\log(n))$.
Now we have to bound $Var(\Gamma)$. It suffices to bound $\mathbb{E}[\Gamma^2]$. We have :
$$\begin{aligned}\mathbb{E}[\Gamma^2] &\le 4m^4\sum_{1\le j<j'\le m}(a_j)^2(a_{j'})^2\mathbb{E}\left[\left(\sum_{k=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j\rfloor}\sum_{k'=\lfloor nt_{j'-1}\rfloor+1}^{\lfloor nt_{j'}\rfloor}\mathbf{1}_{\{S_k=S_{k'},\ \mathcal{I}_k=\mathcal{I}_{k'}\}}\right)^2\right]\\ &\le 4m^4\sum_{1\le j<j'\le m}a_ja_{j'}\sum_{0\le k_1\le k_2<k_1'\le k_2'\le\lfloor nt_m\rfloor}\mathbb{P}(\{S_{k_1}=S_{k_1'}\text{ and }S_{k_2}=S_{k_2'}\}).\end{aligned}$$
In the proof of proposition 6, we estimate this quantity and prove that it is in $O(n^2)$ (see the term A_2 in section 2.4), *qed.*

This ends the proof of the convergence of the finite dimensional distributions (for the probability measure $(h\nu)\otimes\mathbb{P}_0$).

2. **Tightness.** Let us notice that, once $(x,\ell_0)\in\bar{M}\times\mathbb{Z}^2$ fixed, the distribution of $(Z_n(\Pi_0(x,\ell_0),\cdot))_n$ with respect to $\mathbb{P}_0$ does not depend on ℓ_0 in $\mathbb{Z}^2$. Hence, the tightness with respect to $(h\nu)\otimes\mathbb{P}_0$ follows from the tightness for $\bar{\nu}\otimes\mathbb{P}_0$.

   Following [3] and, according to a lemma of [2] (page 88), let us prove that, for any $\varepsilon>0$, there exists $\lambda>0$ such that, if n is large enough, then we have :
$$\mathbb{P}(\sup_{i\le n}|Z_i|\ge\lambda\sqrt{n\log(n)})\le\frac{\varepsilon}{\lambda^2}.$$
   Let $\varepsilon>0$. For any $n\ge 1$, let us define : $Z_n^*:=\max_{i=0,...,n}Z_i$. Let us recall the general argument given by Bolthausen ([3] pp. 114-115). We will be able to use this argument since $Var(Z_m)=O(m\log(m))$ and since $\frac{V_m}{m\log(m)}$ converges in probability to some positive constant. Let any real number $\rho>\sqrt{2}$ and any integer $m\ge 1$. For any $(x,\ell_0)\in\bar{M}\times\mathbb{Z}^2$, using the fact that $Var_{\mathbb{P}_0}[Z_m(\Pi_0(x,\ell_0),\cdot)]=V_m(x)$, we have :
$$\begin{aligned}\mathbb{P}_0(Z_m^*(\Pi_0(x,\ell_0),\cdot)\ge\rho\sqrt{V_m(x)}) &\le \mathbb{P}_0(Z_m(\Pi_0(x,\ell_0),\cdot)\ge(\rho-\sqrt{2})\sqrt{V_m(x)})+\\ &\quad+\mathbb{P}_0\left(Z_{m-1}^*(\Pi_0(x,\ell_0),\cdot)\ge\rho\sqrt{V_m(x)}\right)\times\\ &\quad\times\mathbb{P}_0\left(Z_{m-1}^*(\Pi_0(x,\ell_0),\cdot)-Z_m(\Pi_0(x,\ell_0),\cdot)\ge\sqrt{2}\sqrt{V_m(x)})\right)\\ &\le \mathbb{P}_0\left(Z_m(\Pi_0(x,\ell_0),\cdot)\ge(\rho-\sqrt{2})\sqrt{V_m(x)}\right)+\frac{1}{2}\mathbb{P}_0\left(Z_m^*(\Pi_0(x,\ell_0),\cdot)\ge\rho\sqrt{V_m(x)}\right).\end{aligned}$$
   (see [3], this comes from an argument of [17]). The same holds if we replace Z_n by $-Z_n$. Hence, we have :
$$(\bar{\nu}\otimes\mathbb{P}_0)\left(\max_{j=1,...,m}|Z_j|\ge\rho\sqrt{\sigma^2V_m}\right)\le 2(\bar{\nu}\otimes\mathbb{P}_0)\left(|Z_m|\ge(\rho-\sqrt{2})\sqrt{\sigma^2V_m}\right).$$

But, from proposition 8, we know that $\frac{V_m}{m\log(m)}$ converges in probability to some constant $a>0$. Let $\varepsilon>0$. Let us take $\delta=\frac{\varepsilon}{\lambda^2}$. There exists $m_1(\delta)$ such that, if $m\ge m_1(\delta)$, then we have : $\bar\nu\left(\{V_m<\frac{am\log(m)}{2}\}\right)<\frac{\delta}{6}$ and $\bar\nu\left(\{V_m>2am\log(m)\}\right)<\frac{\delta}{6}$. Hence, for all $\rho>2\sqrt{2}$ and all $m\ge m_1(\delta)$, we have :

$$(\bar\nu\otimes\mathbb{P}_0)\left(\max_{j=1,...,m}|Z_j|\ge\rho\sqrt{\sigma^2 am\log(m)}\right)\le\frac{\varepsilon}{6}+(\bar\nu\otimes\mathbb{P}_0)\left(\max_{j=1,...,m}|Z_j|\ge\frac{1}{\sqrt2}\rho\sqrt{\sigma^2V_m}\right)$$

$$\le 2(\bar\nu\otimes\mathbb{P}_0)\left(|Z_m|\ge\frac{1}{\sqrt2}(\frac{\rho}{\sqrt2}-\sqrt2)\sqrt{a\sigma^2 m\log(m)}\right)+\frac{\varepsilon}{2\lambda^2}$$

$$\le\frac{\varepsilon}{\lambda^2},$$

if m is big enough (since $Var(Z_m)=O(m\log(m))$), *qed.*

2.3 Proof of corollary 2

Let us write $Z_t^{(n)}:=\sqrt{\frac{c}{n\log(n)}}Z_{\lfloor nt\rfloor}$ and $Z^{(n)}=(Z_t^{(n)})_{t\ge0}$. According to theorem 1, $Z^{(n)}$ converges in distribution to the Wiener process W in the sense of $D([0;+\infty))$. We will use :

$$y\in M,\ \ u\ge0,\ \ \tilde n(u,y):=\max\left\{n\ge1\ :\ \sum_{k=0}^{n-1}\tau\circ T^k(y)\le u\right\}.$$

For all $(q,\vec v)\in M$ and all $s\in[0;\tau(q,\vec v)[$, we have :

$$\tilde Z_t(q+s\vec v,\vec v)=Z_{\tilde n(t+s,(q,\vec v))}(q,\vec v).$$

Let us define : $\varphi_n(t):=\frac{\tilde n(nt,\cdot)}{n}$. According to section 1.1, it suffices to prove that $\left(Z^{(n)}_{\varphi_n(t)}\right)_{t\ge0}$ converges in distribution to $\frac{W}{\int_{\bar M}\tau\,d\bar\nu}$ in $D([0;+\infty))$ (for the measure $h\nu\otimes\mathbb{P}_0$ with $h(y):=\int_0^{\tau(y)}g(\Delta(y,s))\,ds$). We know that $\left(\frac{\tilde n(nt,\cdot)}{n}\right)_{t\ge0}$ converges in probability to $\left(\frac{t}{\int_{\bar M}\tau\,d\bar\nu}\right)_{t\ge0}$ (in $\mathcal{D}([0;+\infty))$ withe respect to $\bar\nu$ (see, for example, [20]). Since, $\tilde n(u,\Pi_0(x,\ell_0))$ does not depend on ℓ_0, the same convergence in probability holds with respect to $h\nu\otimes\mathbb{P}_0$. This ends our proof, according to a classical argument (see [2] p. 151).

2.4 Proof of proposition 6

We will use the following formula :

$$Var(V_n)=4\sum_{0\le k_1<\ell_1\le n-1}\ \sum_{0\le k_2<\ell_2\le n-1}[\bar\nu(S_{k_1}=S_{\ell_1},\mathcal{I}_{k_1}=\mathcal{I}_{\ell_1},\ S_{k_2}=S_{\ell_2}\text{ and }\mathcal{I}_{k_2}=\mathcal{I}_{\ell_2})$$

$$-\bar\nu(S_{k_1}=S_{\ell_1}\text{ and }\mathcal{I}_{k_1}=\mathcal{I}_{\ell_1})\bar\nu(S_{k_2}=S_{\ell_2}\text{ and }\mathcal{I}_{k_2}=\mathcal{I}_{\ell_2})].$$

Let us define the event $E_{k,l}:=\{S_k=S_\ell\text{ and }\mathcal{I}_k=\mathcal{I}_\ell\}$. The variance of V_n can be rewritten $8A_1+8A_2+8A_3+4A_4$ with :

$$A_1:=\sum_{0\le k_1<\ell_1\le k_2<\ell_2\le n-1}[\bar\nu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2})-\bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2})],$$

$$A_2 := \sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n-1} \left[\bar\nu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2})\right],$$

$$A_3 := \sum_{0\le k_1<k_2<\ell_2\le\ell_1\le n-1} \left[\bar\nu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2})\right],$$

$$A_4 := \sum_{0\le k<\ell\le n-1} \left[\bar\nu(E_{k,\ell}) - (\bar\nu(E_{k,\ell}))^2\right].$$

- Control of A_1.

  Let $0\le k_1<\ell_1\le k_2<\ell_2\le n-1$. According to proposition 3, we have :

$$|\bar\nu(E_{k_1,\ell_1}\cap E_{k_2,\ell_2}) - \bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_1,\ell_1})| \le \frac{C{\tau_1}^{k_2-\ell_1}}{(\ell_1-k_1)(\ell_2-k_2)}.$$

  Hence : $|A_1| = O(n\log^2(n))$.

- Control of A_2.

  – Let us start by the control of the product of the probabilities. We have : $\sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n-1}\bar\nu(E_{k_1,\ell_1})\bar\nu(E_{k_2,\ell_2}) \le C\sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n}\frac{1}{\ell_1-k_1}\frac{1}{\ell_2-k_2}$

$$\le \sum_{\substack{m_1,m_2\ge 0\\ m_3,m_4\ge 1\\ m_1+m_2+m_3+m_4\le n}} \frac{C}{(m_2+m_3)(m_3+m_4)}$$

$$\le \sum_{\substack{m_2\ge 0\\ m_3,m_4\ge 1\\ m_2+m_3+m_4\le n}} \frac{C(n-(m_2+m_3+m_4)+1)}{(m_2+m_3)(m_3+m_4)}$$

$$\le \sum_{k,\ell=1}^{n}\ \sum_{\max(1,k+\ell-n)\le m_3\le\min(k,\ell)} \frac{C(n-(k+\ell-m_3)+1)}{k\ell}$$

$$\le 2\sum_{k=1}^{n}\sum_{\ell=1}^{k}\frac{C(n-k+1)\ell}{k\ell} \le 2\sum_{k=1}^{n}C(n-k+1) = O(n^2).$$

  – Now it suffices to estimate :

$$\sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n-1} \bar\nu(S_{\ell_1}-S_{k_1}=0 \ \text{ and } \ S_{\ell_2}-S_{k_2}=0).$$

  For any choice of $0\le k_1\le k_2<\ell_1<\ell_2\le n$, we have : $\bar\nu(S_{\ell_1}-S_{k_1}=0 \ \text{ and } \ S_{\ell_2}-S_{k_2}=0) = \sum_x \bar\nu(S_{k_2}-S_{k_1}=x,\ S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x)$. The sum is taken over $x\in\mathbb{Z}^2$ such that $|x|\le\|\Phi\|_\infty\min(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)$.

  According to proposition 4, we have :

$$\bar\nu(S_{k_2}-S_{k_1}=x,\ S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x) \le A+B$$

with

$$A := K_0 e^{-\frac{1}{2(k_2-k_1)}a_0\langle x,x\rangle} \frac{\bar\nu(S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x)}{k_2-k_1}$$

$$B := \frac{(\bar\nu(S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x)))^{1/p}}{(k_2-k_1)^{3/2}}.$$

Analogously, we have :

$$\bar\nu(S_{\ell_1}-S_{k_2}=-x \text{ and } S_{\ell_2}-S_{\ell_1}=x) \le A'+B'$$

with

$$A' := K_0 e^{-\frac{1}{2(\ell_1-k_2)}a_0\langle x,x\rangle} \frac{\bar\nu(S_{\ell_2}-S_{\ell_1}=x)}{\ell_1-k_2} \quad \text{and} \quad B' := (\bar\nu(\frac{S_{\ell_2}-S_{\ell_1}=x)))^{1/p}}{(\ell_1-k_2)^{3/2}}.$$

In the same way, we have :

$$\bar\nu(S_{\ell_2}-S_{\ell_1}=x) \le A''+B''$$

with :

$$A'' := K_0 e^{-\frac{1}{2(\ell_2-\ell_1)}a_0\langle x,x\rangle} \frac{1}{\ell_2-\ell_1} \quad \text{and} \quad B'' := \frac{K_0}{(\ell_2-\ell_1)^{3/2}}.$$

* Terms with (A, A', A'').

$$\sum_x \sum_{0\le k_1\le k_2<\ell_1<\ell_2\le n-1} e^{-\left[\frac{1}{2(k_2-k_1)}+\frac{1}{2(\ell_1-k_2)}+\frac{1}{2(\ell_2-\ell_1)}\right]a_0\langle x,x\rangle} \frac{1}{(k_2-k_1)(\ell_1-k_2)(\ell_2-\ell_1)}$$

$$\le c \sum_{k_1,k_2,\ell_1,\ell_2} \frac{\min(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)}{(k_2-k_1)(\ell_1-k_2)(\ell_2-\ell_1)} \le 6cn \sum_{k=1}^{n} \sum_{\ell\le p\le k} \frac{1}{kp} = O(n^2).$$

* Terms with (B, B', B'').
  The sum of these terms over x and over k_1, k_2, ℓ_1, ℓ_2 is less than :

$$cn \sum_{k+\ell+m\le n} \frac{\min(k^2, l^2, m^2)}{k^{3/2}\ell^{3/(2p)}m^{3/(2p^2)}} \le 6cn \sum_{1\le k,\ell,m\le n} \frac{k^{2/3}\ell^{2/3}m^{2/3}}{k^{3/(2p^2)}\ell^{3/(2p^2)}m^{3/(2p^2)}}.$$

  This is in $O(n^2)$ since $1 < p < \frac{3}{2\sqrt{2}}$.
* The remaining terms correspond to (A, A', B''), (A, B', A''), (A, B', B''), (B, A', A''), (B, A', B'') and (B, B', A''). The sum over x and over k_1, k_2, ℓ_1, ℓ_2 of these terms is less (up to some fixed multiplicative constant) than :

$$n \sum_{k+\ell+m\le n} \frac{\min(k,\ell,m^2)}{k^{1/p}\ell^{1/p}m^{3/2}} + n \sum_{k+\ell+m\le n} \frac{\min(k,\ell^2,m^2)}{k^{1/(p^2)}\ell^{3/2}m^{3/(2p)}}.$$

  (The first term corresponds to (A, A', B''), (A, B', A''), (B, A', A'') and the second one to the others).
  The first term is in $O(n^{4-2/p}) = O(n^2)$ since $p < 3/2$. Indeed we have :

$$n \sum_{1\le k,\ell,m\le n} \frac{\min(k,\ell,m^2)}{k^{1/p}\ell^{1/p}m^{3/2}} \quad \le \quad n \sum_{1\le k,\ell,m\le n} \frac{k^{1/p-1/2}\ell^{1/p-1/2}m^{2-2/p}}{k^{1/p}\ell^{1/p}m^{3/2}} = O(n^2),$$

since $1 < p < 4/3$. Now let us estimate the second term :

$$n \sum_{k+\ell+m \le n} \frac{\min(k, \ell^2, m^2)}{k^{1/(p^2)} \ell^{3/2} m^{3/(2p)}} \le n \sum_{k+\ell+m \le n} \frac{k^{1-1/(2p^2)} \ell^{2(1/(4p^2))} m^{2(1/(4p^2))}}{k^{1/(p^2)} \ell^{3/2} m^{3/(2p)}} \le O(n^2),$$

since $1 < p^2 < 3/2$.

- Control of A_3.
  - First we have :

$$\sum_{0 \le k_1 < k_2 < \ell_2 < \ell_1 \le n-1} \bar{\nu}(S_{k_1} = S_{\ell_1}) \bar{\nu}(S_{k_2} = S_{\ell_2}) \le C \sum_{0 \le k_1 < k_2 < \ell_2 < \ell_1 \le n-1} \frac{1}{\ell_1 - k_1} \frac{1}{\ell_2 - k_2}$$

$$\le C \sum_{p=3}^{n-1} \sum_{\ell=1}^{p-2} \frac{n-p}{p} \frac{p-\ell}{\ell} \le C \sum_{p=3}^{n-1} \sum_{\ell=1}^{p-2} \frac{n}{\ell} \le O(n^2 \log(n)).$$

  - We have to estimate :

$$\sum_{0 \le k_1 < k_2 < \ell_2 < \ell_1 \le n-1} \bar{\nu}(S_{k_1} = S_{\ell_1} \text{ and } S_{k_2} = S_{\ell_2}).$$

Let us write :

$$\bar{\nu}(S_{k_1} = S_{\ell_1} \text{ and } S_{k_2} = S_{\ell_2}) = \sum_x \bar{\nu}(S_{k_2} - S_{k_1} = x,\ S_{\ell_2} - S_{k_2} = 0 \text{ and } S_{\ell_2} - S_{\ell_1} = -x).$$

The sum is taken over the x in $\mathbb{Z}^2$ such that : $|x|_\infty \le \|\Phi\|_\infty \min(k_2 - k_1, \ell_2 - k_2, \ell_1 - \ell_2)$. To estimate this quantity, we use proposition 4. We have :

$$\bar{\nu}(S_{k_1+m} - S_{k_1} = x,\ S_{k_1+m} - S_{k_1+m+k} = 0 \text{ and } S_{k_1+m+k+\ell} - S_{k_1+m+k} = -x) \le A + B + C,$$

with :

$$A := \frac{e^{-\frac{a_0 \langle x, x\rangle}{2m}} \bar{\nu}(S_{k_1+m} - S_{k_1+m+k} = 0 \text{ and } S_{k_1+m+k+\ell} - S_{k_1+m+k} = -x)}{m},$$

$$B := \frac{e^{-\frac{a_0 \langle x, x\rangle}{2m}} \bar{\nu}(S_{k_1+m} - S_{k_1+m+k} = 0 \text{ and } S_{k_1+m+k+\ell} - S_{k_1+m+k} = -x)^{1/p}}{m^{3/2}},$$

$$C := \frac{\bar{\nu}(S_{k_1+m} - S_{k_1+m+k} = 0 \text{ and } S_{k_1+m+k+\ell} - S_{k_1+m+k} = -x)^{1/p}}{m^2}.$$

Moreover, we have :

$$\bar{\nu}(S_{k_1+m} - S_{k_1+m+k} = 0 \text{ and } S_{k_1+m+k+\ell} - S_{k_1+m+k} = -x) \le A' + B'$$

with

$$A' := \frac{\bar{\nu}(S_{k_1+m+k+\ell} - S_{k_1+m+k} = -x)}{k},$$

$$B' := \frac{\bar{\nu}(S_{k_1+m+k+\ell} - S_{k_1+m+k} = -x)^{1/p}}{k^{3/2}}.$$

Finally we have :

$$\bar{\nu}(S_{k_1+m+k+\ell} - S_{k_1+m+k} = -x) \leq A'' + B'',$$

with

$$A'' := \frac{e^{-\frac{a_0\langle x,x\rangle}{2\ell}}}{\ell} \quad \text{and} \quad B'' := \frac{1}{\ell^{3/2}}.$$

1. The term with (A, A', A'') is less than : $\frac{e^{-\frac{a_0\langle x,x\rangle}{2m}} e^{-\frac{a_0\langle x,x\rangle}{2\ell}}}{mk\ell}$.
   The sum of these quantities over x, m, k, ℓ, is less than :

$$O\left(n \sum_{m,k,\ell} \frac{\min(m, k^2, \ell)}{km\ell}\right) \leq O\left(n \sum_{m,k,\ell} \frac{m^{1/2}\ell^{1/2}}{km\ell}\right) = O(n^2 \log(n))$$

2. The term with (A, A', B'') is less than : $\frac{e^{-\frac{a_0\langle x,x\rangle}{2m}}}{mk\ell^{3/2}}$.
   The sum over x, m, k, ℓ is less than : $n\sum_{m,k,\ell} \frac{m}{mk\ell^{3/2}} = O(n^2 \log(n))$.

3. The terms with (A, B', A'') and (B, B', A'') are less than $\frac{e^{-\frac{a_0\langle x,x\rangle}{2m}} e^{\frac{-a_0\langle x,x\rangle}{2\ell p^2}}}{mk^{3/(2p)}\ell^{1/(p^2)}}$. The sum of these terms over x, k, m, ℓ is less than :

$$O\left(n \sum_{m,k,\ell} \frac{\min(m, k^2, \ell)}{mk^{3/(2p)}\ell^{1/(p^2)}}\right) \leq n \sum_{m,k,\ell} \frac{m^{1/2}(k^2)^{1-\frac{1}{p^2}}\ell^{\frac{1}{p^2}-\frac{1}{2}}}{mk^{3/(2p)}\ell^{1/(p^2)}} = O(n^2).$$

   Indeed, since $p^2 < \frac{7}{6}$, we have : $2 - \frac{7}{2p^2} < -1$.

4. The terms with (A, B', B'') and with (B, B', B'') is less than : $\frac{e^{-\frac{a_0\langle x,x\rangle}{2m}}}{mk^{3/(2p)}\ell^{3/(2p^2)}}$. The sum of these quantities over x, m, k, ℓ is less than :

$$O\left(n \sum_{m,k,\ell} \frac{m}{mk^{3/(2p)}\ell^{3/(2p^2)}}\right) \leq O\left(n \sum_{m,k,\ell} \frac{1}{k^{3/(2p)}\ell^{3/(2p^2)}}\right) = O(n^2),$$

   since $p^2 < 3/2$.

5. The term with (B, A', A'') is less than $\frac{e^{-\frac{a_0\langle x,x\rangle}{2m}} e^{-\frac{a_0\langle x,x\rangle}{2\ell p}}}{m^{3/2}k^{1/p}\ell^{1/p}}$. The sum of these terms over (x, m, k, ℓ) is in :

$$O\left(n \sum_{m,k,\ell} \frac{\min(m, k^2, \ell)}{m^{3/2}k^{1/p}\ell^{1/p}}\right) \quad \leq \quad O\left(n \sum_{m,k,\ell} \frac{m^{(1/4)-(3/(2p))}(k^2)^{\frac{1}{2p}-\frac{1}{4}}\ell^{(1/p)-(1/2)}}{m^{3/2}k^{1/p}\ell^{1/p}}\right).$$

   Since $p < 2$, this sum is in $O(n^2)$.

6. The term with (B, A', B'') is less than $\frac{e^{-\frac{a_0\langle x,x\rangle}{2m}}}{m^{3/2}k^{1/p}\ell^{3/(2p)}}$. The sum of these terms over (x, m, k, ℓ) is in $O\left(n\sum_{m,k,\ell} \frac{\min(m,k^2,\ell^2)}{m^{3/2}k^{1/p}\ell^{3/(2p)}}\right)$ and hence in :

$$O\left(n \sum_{m,k,\ell} \frac{m^{1/(2p)}(k^2)^{1-(1/(2p))-(1/(4p^2))}, (\ell^2)^{1/(4p^2)})}{m^{3/2}k^{1/p}\ell^{3/(2p)}}\right) = O(n^2),$$

   since $1 < p < 5/4$ implies $\frac{1}{2p} - \frac{3}{2} < -1$ and $2 - \frac{2}{p} - \frac{1}{2p^2} < 0$ and $\frac{1}{2p^2} - \frac{3}{2p} < -1$.

7. The term in (C, A', A'') is less than $\frac{e^{-\frac{a_0\langle x,x\rangle}{2\ell p}}}{m^2 k^{\frac{1}{p}}\ell^{\frac{1}{p}}}$.
The sum of these terms over (x, m, k, ℓ) is in :

$$O\left(n\sum_{m,k,\ell}\frac{\min(m^2,k^2,\ell)}{m^2k^{\frac{1}{p}}\ell^{\frac{1}{p}}}\right)=O\left(n\sum_{m,k,\ell}\frac{(m^2)^{\frac{1}{2p}}(k^2)^{\frac{1}{3}-\frac{1}{6p}}\ell^{\frac{2}{3}-\frac{1}{3p}}}{m^2k^{\frac{1}{p}}\ell^{\frac{1}{p}}}\right).$$

This is in $O(n^2)$ since $1 < p < \frac{8}{7}$ implies $\frac{1}{p} - 2 < -1$ and $\frac{2}{3} - \frac{4}{3p} < -\frac{1}{2}$.

8. The term with (C, A', B'') is less than $\frac{1}{m^2k^{1/p}\ell^{3/(2p^2)}}$.
The term with (C, B', A'') is less than : $\frac{e^{-\frac{a_0\langle x,x\rangle}{2\ell p^2}}}{m^2k^{3/(2p)}\ell^{1/(p^2)}}$.
The term with (C, B', B'') is less than $\frac{1}{m^2k^{3/(2p)}\ell^{3/(2p^2)}}$.
The sum of these terms over (x, m, k, ℓ) is in :

$$O\left(n\sum_{m,k,\ell}\frac{\min(m^2,k^2,\ell^2)}{m^2k^{3/(2p^2)}\ell^{1/(p^2)}}\right)\leq O\left(n\sum_{m,k,\ell}\frac{(m^2)^{\frac{1}{2p}}(k^2)^{\frac{3}{4p^2}-\frac{1}{2}}(\ell^2)^{\frac{3}{2}-\frac{1}{2p}-\frac{3}{4p^2}}}{m^2k^{\frac{3}{2p^2}}\ell^{\frac{1}{p^2}}}\right).$$

This is $O(n^2\log(n))$ in since $p^2 < 7/6$.

- Control of A_4. We obviously have $A_4 = O(n^2)$, *qed.*

qed.

3 Proof of proposition 7

Since $\bar\nu(\sup_{\ell\in\mathbb{Z}^2}\mathcal{N}_\ell(n)\geq\varepsilon n^a)\leq(2n\max\tau+1)^2\sup_{\ell\in\mathbb{Z}^2}\frac{\mathbb{E}[(\mathcal{N}_\ell(n))^m]}{\varepsilon^m n^{am}}$, proposition 7 will follow from the following lemma and from the first Borel Cantelli lemma :

Lemma 9 *For all $k \geq 1$ and all $q > 0$, we have : $\sup_{\ell\in\mathbb{Z}}\mathbb{E}[(\mathcal{N}_\ell(n))^k] = o(n^q)$.*

Proof. Let $\ell \in \mathbb{Z}$. We have :

$$\mathbb{E}[(\mathcal{N}_\ell(n))^k]\leq k!\sum_{0\leq j_1\leq\ldots\leq j_k\leq n-1}\bar\nu(S_{j_1}=\ell,S_{j_2}=S_{j_1},\ldots,S_{j_k}=S_{j_{k-1}}).$$

But, according to the proof of proposition 3 (with an easy induction), for all $m \geq 2$, there exists $C_m > 1$ such that, for all $0 \leq k_1 \leq \ldots \leq k_m$, for all $\alpha \in \mathbb{Z}^2$, we have :

$$\bar\nu(S_{k_1}=\alpha,S_{k_2}=S_{k_1},\ldots,S_{k_m}=S_{k_{m-1}})\leq\frac{C_m}{(k_1+1)(k_2-k_1+1)\ldots(k_m-k_{m-1})},$$

qed.

A Proof of the extensions of the local limit theorem

Let $1 < p < 2$. We use the Young towers and a result of Hennion and Hervé [14] (adaptation of an idea of Nagaev [15, 16], see also [13]).

Let us recall some hyperbolic properties of the billiard transformation. For almost $\bar{\nu}$-every point in $\bar{M}$, there exist two unique maximal C^1-curves $\gamma^s(x)$ and $\gamma^u(x)$ such that there exists $\tilde{C} > 0$ and $\tilde{\theta} \in]0;1[$ such that :

- For all integer $n \geq 0$, $\bar{T}^n$ is C^1-regular on a neighbourhood of $\gamma^s(x)$ and the diameter of $\bar{T}^n(\gamma^s(x))$ is bounded from away by $\tilde{C}\tilde{\theta}^n$.
- For all integer $n \geq 0$, $\bar{T}^{-n}$ is C^1-regular on a neighbourhood of $\gamma^u(x)$ and the diameter of $\bar{T}^{-n}(\gamma^u(x))$ is bounded from away by $\tilde{C}\tilde{\theta}^n$.

The curves $\gamma^s(x)$ are called **stable curves** and the curves $\gamma^u(x)$ are called **unstable curves**.

A.1 Young towers

In [26], Young constructs an integer d and two dynamical systems $(\tilde{M}, \tilde{\nu}, \tilde{T})$ and $(\hat{M}, \hat{\nu}, \hat{T})$ such that $(\tilde{M}, \tilde{\nu}, \tilde{T})$ is an extension of $(\bar{M}, \bar{\nu}, \bar{T}^d)$ and of $(\hat{M}, \hat{\nu}, \hat{T})$, i.e. there exist two measurable functions $\pi : (\tilde{M}, \tilde{\nu}, \tilde{T}) \to (\bar{M}, \bar{\nu}, \bar{T}^d)$ and $\hat{\pi} : (\tilde{M}, \tilde{\nu}, \tilde{T}) \to (\hat{M}, \hat{\nu}, \hat{T})$ such that : $\pi \circ \tilde{T} = \bar{T}^d \circ \pi$, $\nu = (\pi)_* \tilde{\nu}$, $\hat{\pi} \circ \tilde{T} = \hat{T} \circ \hat{\pi}$ and $\hat{\nu} = (\hat{\pi})_* \tilde{\nu}$. Let us give some useful details. Young constructs a well chosen set $\Lambda = \left(\bigcup_{\gamma^u \in \Gamma^u_\Lambda} \gamma^u\right) \cap \left(\bigcup_{\gamma^s \in \Gamma^s_\Lambda} \gamma^s\right)$ where Γ^u_Λ is a set of unstable curves of $\bar{T}$ and where Γ^s_Λ is a set of stable curves of $\bar{T}$ such that each $\gamma^s \in \Gamma^s_\Lambda$ meets each $\gamma^u \in \Gamma^u_\Lambda$ at exactly one point. Then she constructs a well chosen return time $R(\cdot)$ for $\bar{T}$ in Λ (not the first return time) such that :

- the set Λ can be decomposed in pairwise disjoint subsets : $\Lambda = \bigcup_{i \geq 0} \Lambda_i$ with $\Lambda_i = \left(\bigcup_{\gamma^u \in \Gamma^u_\Lambda} \gamma^u\right) \cap \left(\bigcup_{\gamma^s \in \Gamma^s_i} \gamma^s\right)$, with $\Gamma^s_i \subseteq \Gamma^s_\Lambda$;
- on Λ_i, the return time R is equal to a constant R_i;
- The map $\bar{T}^{R_i}$ is regular on Λ_i and we have : $\bar{T}^{R_i}(\Lambda_i) = \left(\bigcup_{\gamma^u \in \Gamma^u_i}\right) \cap \left(\bigcup_{\gamma^s \in \Gamma^s_\Lambda}\right)$, with $\Gamma^u_i \subseteq \Gamma^u_\Lambda$.

Let us notice that $\sum_{j=0}^{R(\cdot)-1} \Phi \circ \bar{T}^j$ is equal to some constant L_i on each Λ_i. Using the fact that the billiard system in the plane (M, ν, T) is totally ergodic, we can adapt Young's construction in such a way that $L_0 = (0,0)$, $L_1 = (1,0)$, $L_2 = (0,1)$ and R_1 and R_2 and $R_3 - 1$ are multiples of R_0 (the idea is to adapt the construction of the first four sub-parallelogramms Λ_0, Λ_1, Λ_2 and Λ_3 and then to follow Young's construction detailed in appendix D of [18], cf. appendix C in the present paper for the general ideas).

Now, we take d to be the biggest common divisor of the R_i. With our adaptation, d is equal to 1. This fact is not essential in our proof but it simplifies formulas. We take : $\tilde{M} := \{(x, \ell) : x \in \Lambda,\ \ell \in \mathbb{Z}_+,\ \ell < R(x)\}$ and $\tilde{T}(x, \ell) = (x, \ell+1)$ if $\ell < R(x)-1$ and $\tilde{T}\,(x, R(x)-1) = (\bar{T}(x), 0)$. We do not detail the construction of $\tilde{\nu}$ here. The system $(\hat{M}, \hat{\nu}, \hat{T})$ is obtained from $(\tilde{M}, \tilde{\nu}, \tilde{T})$ by quotienting Λ along the stable curves : $\hat{M} = \{(x, \ell) : x \in \gamma^u_0,\ \ell \in \mathbb{Z}_+,\ \ell < R(x)\}$ for a fixed unstable curve γ_0^u belonging to Γ_Λ^u and with $\hat\pi : (x,\ell) \mapsto (\gamma^s(x)\cap\gamma_0^u,\ell)$. Let us notice that if a measurable function $g : \bar M \to \mathbb{C}$ is constant on the stable curves, then there exists a unique $\hat g : \hat M \to \mathbb{C}$ such that $g\circ\pi = \hat g\circ\hat\pi$. In the following, we will consider the function $\psi : \hat M \to \mathbb{C}$ such that :

$$\psi\circ\hat\pi = \Phi\circ\pi.$$

Young defines a separation time $\hat s(\cdot,\cdot)$ on $\hat M$ such that if $\hat s(x,y) > n$, then we have $\hat s(x,y) = n + \hat s(\hat T^n(x),\hat T^n(y))$ and the sets $\pi(\hat\pi^{-1}(\{x\}))$ and $\pi(\hat\pi^{-1}(\{y\}))$ are contained in the same connected component of $\bar M\setminus\bigcup_{j=0}^{(n+1)d-1}\bar T^{-j}(R_0)$. Moreover, if x and y belongs to the same $\hat\pi\left(\Lambda_i\times\{0\}\right)$, then $\hat s(x,y) > \frac{R(x)}{d}$. Young constructs a functional space :

$$\mathcal{V}_{(\beta,\varepsilon)} := \left\{\hat f : \hat M \to \mathbb{C} \text{ measurable}, \ \|\hat f\|_{\mathcal{V}_{(\beta,\varepsilon)}} < +\infty\right\},$$

where $\left\|\hat f\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} := \left\|\hat f\right\|_{(\beta,\varepsilon,\infty)} + \left\|\hat f\right\|_{(\beta,\varepsilon,h)}$ (for some good choices of $\beta\in]0;1[$ and $\varepsilon > 0$ depending on p) with :

$$\left\|\hat f\right\|_{(\beta,\varepsilon,\infty)} := \sup_{\ell\ge 0}\left\|\hat f_{|\hat\Delta_\ell}\right\|_\infty e^{-\ell\varepsilon} \quad\text{and}\quad \left\|\hat f\right\|_{(\beta,\varepsilon,h)} := \sup_{\ell\ge 0}\ \sup_{\hat x,\hat y\in\hat\Delta_\ell,\ \hat x\ne\hat y}\frac{\left|\hat f(\hat x)-\hat f(\hat y)\right|}{\beta^{\hat s(\hat x,\hat y)}}e^{-\ell\varepsilon},$$

where $\hat\Delta_\ell$ is the ℓ^{th} floor of the Tower $\hat M$ (i.e. $\hat\Delta_\ell = \{(x,q)\in\hat M \ : \ q=\ell\}$). Moreover there exists C_0 such that we have : $\|\cdot\|_{L^{\frac{p}{p-1}}(\hat\nu)} \le C_0\|\cdot\|_{\mathcal{V}_{(\beta,\varepsilon)}}$. Let us denote by $\mathcal{V}_{(\beta,0)}$. the set of functions $\hat f : \hat M\to\mathbb{C}$ such that the following quantity is finite : $\|\hat f\|_{\mathcal{V}_{(\beta,0)}} := \left\|\hat f_{|\hat\Delta_\ell}\right\| + \sup_{\ell\ge0}\sup_{\hat x,\hat y\in\hat\Delta_\ell,\ \hat x\ne\hat y}\frac{|\hat f(\hat x)-\hat f(\hat y)|}{\beta^{\hat s(\hat x,\hat y)}}$. It is easy to prove the following :

Lemma 10 *If g belongs to $\mathcal{V}_{(\beta,\varepsilon)}$ and if h belongs to $\mathcal{V}_{(\beta,0)}$, then gh belongs to $\mathcal{V}_{(\beta,\varepsilon)}$ and we have :*

$$\|\hat gh\|_{\mathcal{V}_{(\beta,\varepsilon)}} = \|g\|_{\mathcal{V}_{(\beta,\varepsilon)}}\|h\|_{\mathcal{V}_{(\beta,0)}}.$$

The adjoint operator P of $g\mapsto g\circ\hat T$ on $L^2(\hat m)$ satisfies $P\hat\rho = \hat\rho$ with $\hat\rho$ a bounded function such that $\forall\hat x,\hat y\in\hat M,\quad |h(\hat x)-h(\hat y)|\le c_{\hat\rho}\beta^{\hat s(\hat x,\hat y)}$. There exist $\tau_1\in]0;1[$ and $C_1 > 0$ such that, for any integer $n\ge 0$ and any $\hat f\in\mathcal{V}_{(\beta,\varepsilon)}$ with $\int_{\hat M}\hat f\,d\hat m = 0$, we have $\left\|P^n\hat f\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} \le C_1{\tau_1}^n\left\|\hat f\right\|_{\mathcal{V}_{(\beta,\varepsilon)}}$. The adjoint operator $\hat P$ of $g\mapsto g\circ\hat T$ on $L^2(\hat\nu)$ is given by : $\hat P(g) := \frac{P(\hat\rho g)}{\hat\rho}$. Because of the properties of P and of $\hat\rho$, $\hat P$ is a continuous linear operator on $\mathcal{V}_{(\beta,\varepsilon)}$ satisfying $\hat P\mathbf{1} = \mathbf{1}$ and there exist two real numbers $C_2 > 0$ and $\tau_2\in]0;1[$ such that, for all integer $n\ge 0$ and for all $\hat f\in\mathcal{V}_{(\beta,\varepsilon)}$ such that $\int_{\hat M}\hat f\,d\hat\nu = 0$, we have : $\left\|\hat P^n\hat f\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} \le C_2{\tau_2}^n\left\|\hat f\right\|_{\mathcal{V}_{(\beta,\varepsilon)}}$. Moreover there exist inverse branches β of $\hat T$ and a function κ such that $\hat P(g)(\hat x) = \sum_\beta\kappa(\beta(x))g(\beta(\hat x))$. The inverse branches are such that,

$$\text{if } \hat s(\hat x,\hat y)\ge 1, \text{ then } \hat s(\beta(\hat x),\beta(\hat y)) = 1+\hat s(\hat x,\hat y). \tag{2}$$

There exists a constant c_κ and $\alpha\in]0;1[$ such that, for all $\hat x$ and $\hat y$, we have :

$$0 < \kappa(\hat x)\le 1 \text{ and } \left|\log\left(\frac{\kappa(\hat x)}{\kappa(\hat y)}\right)\right| \le c_\kappa\beta^{\hat s(\hat x,\hat y)}. \tag{3}$$

In the sequel, it will be very useful to work with the following operators family $(\hat P_u)_{u\in\mathbb{R}^2}$ given by $\hat P_u(h) := \hat P(\exp(i < u,\psi >)h)$.

A.2 Transfer Operator

Let $\mathcal{B}$ be any complex Banach space. We define the set $\mathcal{B}'$ of continuous $\mathbb{C}$-linear maps from $\mathcal{B}$ in $\mathbb{C}$. We endow this set with the norm $\|\cdot\|_{\mathcal{B}'}$ given by : $\|A\|_{\mathcal{B}'} := \sup_{\|f\|_{\mathcal{B}}=1} |A(f)|$. We denote by $\mathcal{L}_{\mathcal{B}}$ the set of continuous $\mathbb{C}$ -linear endomorphisms of $\mathcal{B}$. We endow this set with the norm $\|\cdot\|_{\mathcal{L}_{\mathcal{B}}}$ given by : $\|P\|_{\mathcal{L}_{\mathcal{B}}} := \sup_{\|f\|_{\mathcal{B}}=1} \|P(f)\|_{\mathcal{B}}$.

Theorem A.1 (Multidimensional version of theorem IV.8 of [14],) *Let $\mathcal{B}$ be a complex Banach space. Let U_0 be an open subset of $\mathbb{R}^\ell$ containing $0_{\mathbb{R}^\ell}$. Let $m \geq 1$ be some integer. Let $(Q(t))_{t\in U_0}$ be a family of continuous linear operators on $\mathcal{B}$ such that the application $t \mapsto Q(t)$ is in $C^m(U_0, \mathcal{L}_{\mathcal{B}})$ and such that there exist two subspaces $\mathcal{F}$ and $\mathcal{H}$ of $\mathcal{B}$ with : $\mathcal{B} = \mathcal{F} \oplus \mathcal{H}$ and $Q(0)(\mathcal{F}) \subseteq \mathcal{F}$ and $Q(0)(\mathcal{H}) \subseteq \mathcal{H}$, $\dim(\mathcal{F}) = 1$ and $Q(0)_{|\mathcal{F}} \equiv id_{|\mathcal{F}}$, the spectral radius of $Q(0)_{|\mathcal{H}}$ being strictly less than 1.*

Then there exists an open set U_1 containing 0 and contained in U_0, there exist three real numbers $\eta_1 > 0, \eta_2 > 0$, $c_1 \geq 0$ and four functions $\lambda \in C^m(U_1, \mathbb{C})$, $v \in C^m(U_1, \mathcal{B})$, $\phi \in C^m(U_1, \mathcal{B}')$ and $N \in C^m(U_1, \mathcal{L}_{\mathcal{B}})$ such that, for all $t \in U_1$, we have, for all $n \geq 1$,

$$Q(t)^n(h) = \lambda(t)^n(\phi(t)(h))v(t) + N(t)^n(h),$$

with $Q(t)v(t) = \lambda(t)v(t)$, $Q(t)^*\phi(t) = \lambda(t)\phi(t)$ and $(\phi(t))(v(t)) = 1$ and $|\lambda(t)| \geq 1 - \eta_1$ and, for all $\ell = 0, ..., m$, for all $i_1, ..., i_\ell \in \{1, ..., \ell\}$ and all $n \geq 1$, $\left\| \frac{\partial^\ell}{\partial t_{i_1} \cdots \partial t_{i_\ell}} (N(t)^n) \right\|_{\mathcal{L}_{\mathcal{B}}} \leq c_1(1-\eta_1-\eta_2)^n$.

Idea of the proof. This is the multidimensional version of theorem IV-8 of [14] which is based on the implicit functions theorem (see chapter XIV of [14]), *qed.*

Lemma A.2 *The map $t \mapsto \hat{P}_t$ is in $C^\infty(\mathbb{R}^2, \mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}})$. Moreover, for all $t \in \mathbb{R}^2$, for all integer $m_1 \geq 0$ and $m_2 \geq 0$ with $m_1 + m_2 \geq 1$, we have : $\frac{\partial^{m_1+m_2}}{\partial t_1{}^{m_1} \partial t_2{}^{m_2}} \hat{P}_t = \hat{P}_t(i^{m_1+m_2} \psi_1{}^{m_1} \psi_2{}^{m_2} \cdot) = \hat{P}\left(i^{m_1+m_2} e^{i\langle t, \hat{f}\rangle} \psi_1{}^{m_1} \psi_2{}^{m_2} \cdot\right)$.*

We apply theorem A.1 to $Q(t) = \hat{P}_t$ and $\mathcal{B} = \mathcal{V}_{(\beta,\varepsilon)}$. We have $\lambda(0) = 1$, $v(0) = \mathbf{1}$ and $\phi(0) = \hat{\nu}$. Moreover, since $m \geq 2$, according to corollaries III-11 and III-12 of [14]], we get : $\nabla\lambda(0) = 0$ and $\mathrm{Hess}\lambda(0) = -\Sigma^2$ where Σ^2 is the limit of the covariance matrices sequence $\left(Cov\left(\frac{S_n}{\sqrt{n}}\right)\right)_{n\geq 1}$.

Let $\beta > 0$ be such that $]-\beta; \beta[^2$ is contained in the set U_1 given by the previous theorem. We also suppose that there exists some constant $a > 0$ such that, for all $u \in]-\beta; \beta[^2$, $|\lambda_u| \leq \exp(-a\langle u, u\rangle)$ and $\frac{1}{2}\langle\Sigma^2 u, u\rangle > a\langle u, u\rangle$. Moreover, let us prove the following :

Lemma 11 *The spectral radia of $\left(\hat{P}_t\right)_{t\in[-\pi;\pi]^2 \setminus]-\beta;\beta[^2}$ are uniformly bounded by some constant less strictly than 1.*

Proof. To this purpose, as Szász and Varjú do in [25], we use a result of Aaronson and Denker in [**?**]. According to this result, it suffices to prove that, for all $u \in [-\pi; \pi]^2 \setminus \{(0,0)\}$, $\hat{P}_u$ has no eigenvalue on the unit circle. Let $u = (u_1, u_2) \in [-\pi; \pi]^2$ with $u \neq (0,0)$. Let us suppose that $\hat{P}_u$ has an eigenvalue of modulus 1. We will prove that $u = (0,0)$. First, let us suppose that there exists $f \in \mathcal{V}_{\beta,\varepsilon}$ non identically equal to zero and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that : $\hat{P}(e^{i\langle u,\psi\rangle} f) = \lambda f$. We will use the fact that $\hat{P}$ has the following form :

$$\hat{P}(h)(x) = \sum_{y\ :\ \hat{T}(y)=x} \kappa(y)h(y), \quad \text{with} \quad \sum_{y\ :\ \hat{T}(y)=x} \kappa(y) = 1 \text{ and } \kappa(y) > 0.$$

- Let us prove that the modulus of f is uniformly bounded.

  Since f belongs to $\mathcal{V}_{\beta,\varepsilon}$, we know that f is bounded on each $\hat{\Delta}_q$. Let $q \geq 0$. Using the previous formula of $\hat{P}$ and the definition of $\hat{T}$, we get that, for all $q \geq 1$, $\|f_{|\hat{\Delta}_q}\|_\infty \leq \|f_{|\hat{\Delta}_0}\|_\infty$ and, for all $(x,q) \in \hat{\Delta}_q$, we have : $|f(x,q)| = |f(x,0)|$.

- We prove that $|f|$ is constant.

  Let $(x,0) \in \hat{\Delta}_0$ such that $|f(x,0)| = \max|f|$. Then, from the definition of $\hat{P}$ and an easy induction, if y is such that $\hat{T}^n(y) = x$ for some $n \geq 1$, then we have $|f(y)| = |f(x)|$. Let $(z,0) \in \hat{\Delta}_0$. For any integer $N \geq 0$, let $x_N = (y_N, 0) \in \hat{\Delta}_0$ be such that $\left(\bar{T}^{R(\cdot)}\right)^N (y_N) = x$ and such that, for all $m = 0, ..., N-1$, $\left(\bar{T}^{R(\cdot)}\right)^m (z)$ and $\left(\bar{T}^{R(\cdot)}\right)^m (y_N)$ are in the same $\hat{\Lambda}_i$. Then $\hat{s}((z,0), x_N) \geq \sum_{k=0}^{N-1} R\left(\left(\bar{T}^{R(\cdot)}\right)^k (z)\right)$. Therefore $\lim_{N\to+\infty} |f(x_N)| = |f(z,0)|$. But $|f(x_N)| = |f(x)|$. Hence, $|f|$ is constant.

- We prove that, for any $y \in \hat{M}$, $\exp\left(i\langle u, \psi(y)\rangle\right) = \lambda \frac{f\circ\hat{T}(y)}{f(y)}$.

  For any $x \in \hat{M}$, we have :

$$|f| = |\lambda f(x)| = \left|\sum_{y\ :\ \hat{T}(y)=x} \kappa(y) e^{i\langle u,\psi(y)\rangle} f(y)\right| \leq \sum_{y\ :\ \hat{T}(y)=x} \kappa(y) \left|e^{i\langle u,\psi(y)\rangle} f(y)\right| = |f|.$$

  Hence, for all $x, y \in \hat{M}$ such that $\hat{T}(y) = x$, we have : $e^{i\langle u,\psi(y)\rangle} f(y) = \lambda f(x)$.

- Hence, for any $y \in \hat{M}$ and any integer $n \geq 1$, we have : $\exp\left(i\langle u, \sum_{j=0}^{n-1} \psi(\hat{T}^j(y))\rangle\right) = \lambda^n \frac{f\circ\hat{T}^n(y)}{f(y)}$.

- Let us prove that f is equal to some constant f_0 on $\hat{\Delta}_0$.

  Let y and z be two points of $\hat{\Delta}_0$. For any N, we consider two points y_N and z_N such that : $\hat{T}^{NR_0}(y_N) = y$ and $\hat{T}^{NR_0}(z_N) = z$ and such that, for all $j = 0, ..., N-1$ $\hat{T}^{jR_0}(y_N)$ and $\hat{T}^{jR_0}(z_N)$ belongs to $\hat{\pi}(\Lambda_0 \times \{0\})$. We have :

$$f(y) = f(y_N)\lambda^{-NR_0} \exp\left(i\langle u, \sum_{j=0}^{NR_0-1} \psi\circ\hat{T}^j(y_N)\rangle\right) = f(y_N)\lambda^{-NR_0}\exp\left(i\langle u, NL_0\rangle\right)$$

  and

$$f(z) = f(z_N)\lambda^{-NR_0}\exp\left(i\langle u, NL_0\rangle\right).$$

  Since $\hat{s}(\hat{x},\hat{y}) \geq N$, we have $\lim_{N\to+\infty} |f(y_N) - f(z_N)| = 0$ and therefore $f(y) = f(z)$.

- Conclusion.

  Let x be in $\hat{\pi}(\Lambda_0 \times \{0\})$. We have : $1 = \exp\left(i\langle u, L_0\rangle\right) = \lambda^{R_0}\frac{f_0}{f_0}$ qnd so $\lambda^{R_0} = 1$.

Let y be in $\hat{\pi}(\Lambda_1 \times \{0\})$. We have : $\exp(iu_1) = \exp\left(i\langle u, L_1\rangle\right) = \lambda^{R_1 \frac{f_0}{f_0}}$. Since R_1 is a multiple of R_0, we get : $\exp(iu_1) = 1$, which implies $u_1 = 0$ (u_1 is the first coordinate of u).

In the same way, by taking Λ_2 instead of Λ_1, we conclude that $u_2 = 0$. Hence $u = (0, 0)$, *qed.*

A.3 Proof of proposition 3

Let us notice that, for all integer $n \geq 1$, we have to estimate :

$$Cov\left(\mathbf{1}_{A_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{A_{i'}} \circ \bar{T}^n, \left\{\mathbf{1}_{A_j}\mathbf{1}_{\{S_k=N_2\}}\mathbf{1}_{A_{j'}} \circ \bar{T}^k\right\} \circ \bar{T}^{n+m}\right)$$

with : $A_u = \{\mathcal{I}_0 = u\}$.

Let us recall that we have $\Phi \circ \pi = \psi \circ \hat{\pi}$. Because of the construction of the separation time $\hat{s}$, if $\hat{s}(\hat{x}, \hat{y}) \geq 1$, then $\psi(\hat{x}) = \psi(\hat{y})$. Hence, ψ is in the functional space and its norm is bounded by $2\|\Phi\|_\infty$. We consider : $\hat{S}_m = \sum_{k=0}^{m-1} \psi \circ \hat{T}^k$. Moreover, there exist four measureable subsets $\hat{A}_i$, $\hat{A}_{i'}$, $\hat{A}_j$, $\hat{A}_{j'}$ in $\hat{M}$ such that, for all $u = i, i', j, j'$, we have : $\mathbf{1}_{\hat{A}_u} \circ \hat{\pi} = \mathbf{1}_{A_u} \circ \pi$ and $\mathbf{1}_{\hat{A}_u}$ belongs to $\mathcal{V}_{(\beta,0)}$. We will use the operators $\hat{P}$ and $\hat{P}_u$ defined previously and the fact that :

$$P_u^m(g.h \circ \hat{T}^m) = hP_u^m(g) \quad \text{and} \quad \mathbb{E}_{\hat{\nu}}[(P_u)^m(g)] = \mathbb{E}_{\hat{\nu}}[e^{i\langle u, \hat{S}_m\rangle}g].$$

We have :

$$\begin{aligned}
&\mathbb{E}_{\bar{\nu}}\left[\mathbf{1}_{A_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{A_{i'}} \circ \bar{T}^n, \left\{\mathbf{1}_{A_j}\mathbf{1}_{\{S_k=N_2\}}\mathbf{1}_{A_{j'}} \circ \bar{T}^k\right\} \circ \bar{T}^{n+m}\right] \\
&\quad = \frac{1}{(2\pi)^4}\int_{[-\pi;\pi]^2}\int_{[-\pi;\pi]^2} e^{-i\langle u, N_1\rangle} e^{-i\langle t, N_2\rangle}\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{A}_i} e^{i\langle u, \hat{S}_n\rangle}\mathbf{1}_{\hat{A}_{i'}} \circ \hat{T}^n \times\right. \\
&\qquad\qquad \left. \times \mathbf{1}_{\hat{A}_j} \circ \hat{T}^{n+m} e^{i\langle t, \hat{S}_k\rangle} \circ \hat{T}^{n+m}\mathbf{1}_{\hat{A}_{j'}} \circ \hat{T}^{n+m+k}\right] dudt \\
&\quad = \frac{1}{(2\pi)^4}\int_{[-\pi;\pi]^2}\int_{[-\pi;\pi]^2} e^{-i\langle u, N_1\rangle} e^{-i\langle t, N_2\rangle}\mathbb{E}_{\hat{\nu}}\left[(\mathbf{1}_{\hat{A}_{j'}}) \times\right. \\
&\qquad\qquad \left. \times \hat{P}_t^k\left\{\mathbf{1}_{\hat{A}_j}\hat{P}^m\left\{\mathbf{1}_{\hat{A}_{i'}}\hat{P}_u^n(\mathbf{1}_{\hat{A}_i})\right\}\right\}\right] dudt.
\end{aligned}$$

Lemma 12 *There exists $K > 0$ such that, For any function $\hat{h} \in \mathcal{V}_{(\beta,\varepsilon)}$ and any nonnegative integer k, we have :*

$$\int_{[-\pi;\pi]^2} \|\hat{P}_u^k(h)\|_{\mathcal{V}_{(\beta,\varepsilon)}}\, du \leq K\frac{\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}}{k+1}.$$

Proof. According to theorem A.1 applied to $\hat{P}$, we have :

$$\begin{aligned}
\int_{[-\pi;\pi]^2} \|\hat{P}_u^k(h)\|_{\mathcal{V}_{(\beta,\varepsilon)}}\, du &= \int_{[-b;b]^2} {\lambda_u}^k \mathbb{E}_{\hat{\nu}}[v_u]\phi_u(h)\, du + O(\delta^h\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}) \\
&= O(1)\int_{[-b;b]^2} e^{-ak\langle u,u\rangle}\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\, du + O(\delta^h\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}) \\
&= O(1)\frac{1}{k}\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\int_{[-b\sqrt{k};b\sqrt{k}]^2} e^{-a\langle v,v\rangle}dv + O(\delta^h\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}),
\end{aligned}$$

qed.

We come back to the proof of lemma 3. Let us write : $g_u = \mathbf{1}_{\hat{A}_{i'}}\hat{P}_u^{n'}(\mathbf{1}_{\hat{A}_i})$ We have :

$$\left|Cov\left(\mathbf{1}_{A_i}\mathbf{1}_{\{S_n=N_1\}}\mathbf{1}_{A_{i'}}\circ\bar{T}^n, \left\{\mathbf{1}_{A_j}\mathbf{1}_{\{S_k=N_2\}}\mathbf{1}_{A_{j'}}\circ\bar{T}^k\right\}\circ\bar{T}^{n+m}\right)\right| \leq$$

$$\begin{aligned}
&\leq \frac{1}{(2\pi)^4}\int_{[-\pi;\pi]^2}\int_{[-\pi;\pi]^2}\left|\mathbb{E}_{\hat{\nu}}\left[(\mathbf{1}_{\hat{A}_{j'}})\hat{P}_t^k\left\{\mathbf{1}_{\hat{A}_j}\left(\hat{P}^{m'}(g_u)-\mathbb{E}_{\hat{\nu}}[g_u]\right)\right\}\right]\right|\,dudt\\
&\leq \tilde{K}\frac{1}{k}\int_{[-\pi;\pi]^2}\left\|\hat{P}^{m'}\left(g_u-\mathbb{E}_{\hat{\nu}}[g_u]\right)\right\|_{\mathcal{V}_{(\beta,\varepsilon)}}\,du\\
&\leq \hat{K}\frac{1}{k'+1}{\tau_1}^{m'}\int_{[-\pi;\pi]^2}\left\|\mathbf{1}_{\hat{A}_{i'}}\hat{P}_u^{n'}(\mathbf{1}_{\hat{A}_i})\right\|_{\mathcal{V}_{(\beta,\varepsilon)}}\,du \leq \tilde{K}\frac{{\tau_1}^{m'}}{(k'+1)(n'+1)}.
\end{aligned}$$

This ends the proof of proposition 3.

Let us notice that, since $u\mapsto v_u$, $u\mapsto\phi_u$ and $u\mapsto\lambda_u$ are C^1 (with $\nabla\lambda(0)=0$ and $Hess\lambda(0)=-\Sigma^2$, we have : $\left\|{\lambda_u}^k\mathbb{E}_{\hat{\nu}}[v_u]\phi_u(h)-e^{-\frac{k}{2}\langle\Sigma^2u,u\rangle}\mathbb{E}_{\hat{\nu}}[h]\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} = O\left(|u|e^{-\frac{a}{2}\langle u,u\rangle}\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}\right)$. Hence, modifying slightly the proof of lemma 12, we can get :

$$\left\|\frac{\frac{1}{(2\pi)^2}\int_{[-\pi;\pi]^2}\hat{P}_u^k(h)\,du-\mathbb{E}_{\hat{\nu}}[h]}{k2\pi\sqrt{det(\Sigma^2)}}\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} \leq K\frac{\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}}{k\sqrt{k}}.$$

Hence, according to the proof of proposition 3 (with $m'=0$), this gives : $\mathbb{P}(S_n=0,\ S_{n+k}-S_n=0)\sim_{n,m\to+\infty}\frac{1}{2\pi det(\Sigma^2)nk}$ used by Dolgopyat, Szász and Varjú in [11]. But, for general A and B as in the hypotheses of proposition 4, with this estimation, we only get :

$$\left|\bar{\nu}\left(A\cap\bar{T}^{-(k+r)}(B)\cap\{S_{k+r}-S_r=N\}\right)-\frac{\bar{\nu}(A)\hat{\nu}(B)}{\sqrt{det(\Sigma^2)}2\pi k}e^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}N,N\rangle}\right| = O\left((\bar{\nu}(B)^{1/p}k^{-\frac{3}{2}}\right).$$

This estimation is not sufficient for our purpose.

A.4 Proof of proposition 4

Let A be a subset of $\bar{M}$ union of connected component of $M\setminus\bigcup_{i=0}^r\bar{T}^{-i}(R_0)$. Let an nonnegative integer k. Let a measurable set B such that, if $x\in B$ then $\gamma^s(x)\subseteq B$. There exists a measurable subset $\hat{A}$ of $\hat{M}$ such that $\mathbf{1}_A\circ\pi=\mathbf{1}_{\hat{A}}\circ\hat{\pi}$. Let us prove that $\|\hat{P}^r(\mathbf{1}_{\hat{A}})\|_{\mathcal{V}_{(\beta,\varepsilon)}}$ is uniformly bounded (in r and in A). Let $\hat{x}$ and $\hat{y}$ in $\hat{M}$ be such that : $\hat{s}(\hat{x},\hat{y})\geq 1$. We have :

$$\hat{P}^r(\mathbf{1}_{\hat{A}})(\hat{x})=\sum_{\beta_r}\prod_{k=0}^{r-1}\kappa(\hat{T}^k(\beta_r(\hat{x})))\ \text{ and }\ \hat{P}^r(\mathbf{1}_{\hat{A}})(\hat{y})=\sum_{\beta_r}\prod_{k=0}^{r-1}\kappa(\hat{T}^k(\beta_r(\hat{y}))),$$

where the sum is taken over the inverse branches β_r of $\hat{T}^r$. Since $\hat{s}(\hat{x},\hat{y})\geq 1$, we have :

$$\hat{s}(\beta_r(\hat{x}),\beta_r(\hat{y}))=\hat{s}(\hat{x},\hat{y})+r\geq 1+r.$$

Hence, according to (2), we have : $\left|\log\left(\frac{\prod_{k=0}^{r-1}\kappa(\hat{T}^k(\beta(x)))}{\prod_{k=0}^{r-1}\kappa(\hat{T}^k(\beta(x)))}\right)\right| \leq c_\kappa \frac{\beta^{\hat{s}(\hat{x},\hat{y})}}{1-\beta}$. According to (3), we get :

$$\begin{aligned}\hat{P}^r(\mathbf{1}_{\hat{A}})(\hat{x}) - \hat{P}^r(\mathbf{1}_{\hat{A}})(\hat{y}) &\leq \sum_{\beta_r}\left(\prod_{k=0}^{r-1}\kappa(\hat{T}^k(\beta_r(\hat{x}))) + \prod_{k=0}^{r-1}\kappa(\hat{T}^k(\beta_r(\hat{y})))\right) c_\kappa \frac{\beta^{\hat{s}(\hat{x},\hat{y})}}{1-\beta} \\ &\leq 2c_\kappa \frac{\beta^{\hat{s}(\hat{x},\hat{y})}}{1-\beta}.\end{aligned}$$

Hence, we have : $\|\hat{P}^r(\mathbf{1}_{\hat{A}})\|_{\mathcal{V}_{\beta,\varepsilon}} \leq 2 + 2\frac{c_\kappa}{1-\beta}$.

There exists $\hat{B} \subseteq \hat{M}$ such that we have $\mathbf{1}_B \circ \pi = \mathbf{1}_{\hat{B}} \circ \hat{\pi}$. Let us recall that we have $\sum_{m=0}^{d-1} \Phi \circ \bar{T}^m \circ \pi = \psi \circ \hat{\pi}$. Because of the construction of the separation time $\hat{s}$, if $\hat{s}(\hat{x},\hat{y}) \geq 1$, then $\psi(\hat{x}) = \psi(\hat{y})$. Hence, ψ is in the functional space and its norm is less than $3\|\Phi\|_\infty$. For any integer $k \geq 1$ we have :

$$\begin{aligned}&\bar{\nu}\left(A \cap \bar{T}^{-(k+r)}(B) \cap \{S_{k+r} - S_r = N\}\right) = \\ &= \frac{1}{(2\pi)^2}\int\int_{[-\pi;\pi]^2} e^{-i\langle u,N\rangle}\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{A}}\mathbf{1}_{\hat{B}} \circ \hat{T}^{k+r} e^{i\langle u, \sum_{j=0}^{k-1}\psi\circ\hat{T}^{j+r}\rangle}\right] du \\ &= \frac{1}{(2\pi)^2}\int\int_{[-\pi;\pi]^2} e^{-i\langle u,N\rangle}\mathbb{E}_{\hat{\nu}}\left[\hat{P}_u^k\left(\hat{P}^r(\mathbf{1}_{\hat{A}})\mathbf{1}_{\hat{B}} \circ \hat{T}^k\right)\right] du\end{aligned}$$

with $\hat{P}_u(h) := \hat{P}(\exp(i < u, \psi >)h)$. Since $\hat{P}_u^k(f \circ T^k g) = f\hat{P}_u^k(g)$, we have :

$$\bar{\nu}\left(A \cap \bar{T}^{-(k+r)}(B) \cap \{S_{(k+r)} - S_r = N\}\right) = \frac{1}{(2\pi)^2}\int\int_{[-\pi;\pi]^2} e^{-i\langle u,N\rangle}\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}}\hat{P}_u^k\left(\hat{P}^r(\mathbf{1}_{\hat{A}})\right)\right] du.$$

We will use the fact that : $|\mathbb{E}_{\hat{\nu}}[gh]| \leq C_0\|g\|_{L^p(\hat{\nu})}\|h\|_{\mathcal{V}_{(\beta,\varepsilon)}}$ and that : $\|\mathbf{1}_{\hat{B}}\|_{L^p(\hat{\nu})} = \bar{\nu}(B)^{\frac{1}{p}}$. According to theorem A.1 and to lemma 11, we have :

$$\begin{aligned}&\bar{\nu}\left(A \cap \bar{T}^{-(k+r)}(B) \cap \{S_{(k+r)} - S_r = N\}\right) = \\ &= \frac{1}{(2\pi)^2}\int\int_{]-\beta;\beta[^2} e^{-i\langle u,N\rangle}{\lambda_u}^k\phi_u(\hat{P}^r(\mathbf{1}_{\hat{A}}))\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}}v_u\right] du + O\left(\delta^k\right)\bar{\nu}(B)^{\frac{1}{p}} \\ &= \frac{1}{k}\frac{1}{(2\pi)^2}\int\int_{]-\beta\sqrt{k};\beta\sqrt{k}[^2} e^{-i\langle\frac{u}{\sqrt{k}},N\rangle}{\lambda_{\frac{u}{\sqrt{k}}}}^k\phi_{\frac{u}{\sqrt{k}}}(\hat{P}^r(\mathbf{1}_{\hat{A}}))\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}}v_{\frac{u}{\sqrt{k}}}\right] du + O\left(\delta^k\right)\bar{\nu}(B)^{\frac{1}{p}}.\end{aligned}$$

- Now, let us show that if we replace ${\lambda_{\frac{u}{\sqrt{k}}}}^k$ by $e^{-\frac{1}{2}\langle\Sigma^2 u,u\rangle}$, we introduce an error uniformly bounded by $O(k^{-2}\bar{\nu}(B)^{1/p})$. Let us notice that we have :

$${\lambda_{\frac{u}{\sqrt{k}}}}^k - \exp\left(-\frac{1}{2}\langle\Sigma^2 u,u\rangle\right) = k\exp\left(-\frac{1}{2}\langle\Sigma^2 u,u\rangle\frac{k-1}{k}\right)A_k(u) + B_k(u)$$

with $A_k(u) = \lambda_{\frac{u}{\sqrt{k}}} - e^{-\frac{1}{2}\langle\Sigma^2\frac{u}{\sqrt{k}},\frac{u}{\sqrt{k}}\rangle}$ and $|B_k(u)| \leq \frac{k(k-1)}{2}e^{-a\langle u,u\rangle\frac{k-2}{k}}\left(\lambda_{\frac{u}{\sqrt{k}}} - e^{-\frac{1}{2}\langle\Sigma^2 u,u\rangle}\right)^2$. Since $\lambda_w - \exp\left(-\langle\Sigma^2 w,w\rangle\right) = O(|w|^3)$, we observe that we have :

$$\frac{1}{k}\frac{1}{(2\pi)^2}\sum_L\int\int_{]-\beta\sqrt{k};\beta\sqrt{k}[^2}\left|B_k(u)\phi_{\frac{u}{\sqrt{k}}}(\hat{P}^r(\mathbf{1}_{\hat{A}}))\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}_{m,L}}v_{\frac{u}{\sqrt{k}}}\right]\right| du = O\left(\frac{1}{k^2}\bar{\nu}(B)^{\frac{1}{p}}\right).$$

We will approximate $A_k(u)$ by $A'_k(u) := \frac{1}{6}\sum_{i,j,j'} \frac{\partial^3 \lambda}{\partial u_i \partial u_j \partial u_{j'}}(0)\frac{u_i u_j u_{j'}}{k^{3/2}}$. We have : $|A_k(u) - A'_k(u)| \le C\frac{|u|^4}{k^2}$. Hence, we have :

$$\frac{1}{(2\pi)^2}\int\int_{]-\beta\sqrt{k};\beta\sqrt{k}[^2} \left| e^{-\frac{1}{2}\langle \Sigma^2 u,u\rangle \frac{k-1}{k}}(A_k(u) - A'_k(u))\phi_{\frac{u}{\sqrt{k}}}(\hat{P}^r(\mathbf{1}_{\hat{A}}))\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}} v_{\frac{u}{\sqrt{k}}}\right]\right| du =$$

$$\le O(k^{-2})\bar{\nu}(B)^{\frac{1}{p}}.$$

Now, we notice that we have :

$$\frac{1}{(2\pi)^2}\int\int_{]-\beta\sqrt{k};\beta\sqrt{k}[^2} e^{-\frac{1}{2}\langle \Sigma^2 u,u\rangle \frac{k-1}{k}} A'_k(u)\left|\phi_{\frac{u}{\sqrt{k}}}(\hat{P}^r(\mathbf{1}_{\hat{A}}))\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}} v_{\frac{u}{\sqrt{k}}}\right] - \bar{\nu}(A)\bar{\nu}(B)\right| du$$

$$= O(k^{-2})\bar{\nu}(B)^{1/p}$$

and :

$$\frac{1}{(2\pi)^2 k^{3/2}}\int\int_{]-\beta\sqrt{k};\beta\sqrt{k}[^2} e^{-i\langle \frac{u}{\sqrt{k}},N\rangle} e^{-\frac{1}{2}\langle \Sigma^2 u,u\rangle \frac{k-1}{k}}\bar{\nu}(A)\bar{\nu}(B)u_i u_j u_{j'}\, du =$$

$$\begin{aligned} &= \frac{\nu(A)\nu(B)}{(2\pi)^2 k^{3/2}}\int\int_{\mathbb{R}^2} e^{-i\langle \frac{u}{\sqrt{k}},N\rangle} e^{-\frac{1}{2}\langle \Sigma^2 u,u\rangle \frac{k-1}{k}} u_i u_j u_{j'}\, du + O\left(\frac{\bar{\nu}(A)\bar{\nu}(B)}{k^2}\right) \\ &= \frac{\nu(A)\nu(B)}{(2\pi)^2 k^{3/2}}\frac{1}{i}\frac{\partial^3 \Phi_k}{\partial u_i \partial u_j \partial u_{j'}}\left(\frac{N}{\sqrt{k}}\right) + O\left(\frac{\bar{\nu}(A)\bar{\nu}(B)}{k^2}\right)\end{aligned}$$

with

$$\Phi_k(X) = \int\int_{\mathbb{R}^2} e^{-i\langle u,X\rangle} e^{-\frac{1}{2}\langle \Sigma^2 u,u\rangle \frac{k-1}{k}}\, du.$$

Hence we have :

$$\Phi_k(X) := \frac{2\pi}{\sqrt{det(\Sigma^2)}}\left(\frac{k}{k-1}\right)\exp\left(-\frac{1}{2}\langle\frac{k}{k-1}(\Sigma^2)^{-1}X,X\rangle\right).$$

Since we have :

$$\left|\frac{\partial^3}{\partial u_i \partial u_j \partial u_{j'}}\Phi_k(X)\right| \le C(\|X\|_2 + \|X\|_2^3)\exp^{-\frac{1}{2}\langle(\Sigma^2)^{-1}X,X\rangle},$$

we get :

$$\frac{1}{(2\pi)^2 k^{3/2}}\sum_L \int\int_{]-\beta\sqrt{k};\beta\sqrt{k}[^2} e^{-i\langle \frac{u}{\sqrt{k}},N\rangle} e^{-\frac{1}{2}\langle(\Sigma^2)^{-1}X,X\rangle \frac{k-1}{k}}\bar{\nu}(A)\bar{\nu}(B)u_i u_j u_{j'}\, du =$$

$$= O\left(\frac{\bar{\nu}(A)\bar{\nu}(B)}{k^{3/2}}\left(\frac{\|N\|}{\sqrt{k}} + \frac{\|N\|^3}{k^{3/2}}\right)e^{-\frac{a}{k}\langle(\Sigma^2)^{-1}N,N\rangle}\right).$$

Let us notice that this quantity is null if N is null and is in $O\left(\frac{\bar{\nu}(A)\bar{\nu}(B)}{k^2}\right)$ if N is fixed but here we want an estimation with constant independent of N.

- Hence it remains to estimate :

$$\frac{1}{k}\frac{1}{(2\pi)^2}\int\int_{]-\beta\sqrt{k};\beta\sqrt{k}[^2} e^{-i\langle \frac{u}{\sqrt{k}},N\rangle} e^{-\frac{1}{2}\langle \Sigma^2 u,u\rangle}\phi_{\frac{u}{\sqrt{k}}}(\hat{P}^r(\mathbf{1}_{\hat{A}}))\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}} v_{\frac{u}{\sqrt{k}}}\right] du.$$

- Using a Taylor expansion, we observe that if we replace : $\phi_{\frac{u}{\sqrt{k}}}(\hat{P}^r(\mathbf{1}_{\hat{A}}))\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}}v_{\frac{u}{\sqrt{k}}}\right]$ by :

$$\begin{aligned} C_k(u) &= \bar{\nu}(A)\bar{\nu}(B) + \\ &+ \left\langle \frac{u}{\sqrt{k}}, \nabla\phi(0)(\hat{P}^r(\mathbf{1}_{\hat{A}}))\bar{\nu}(B) + \hat{\nu}(\hat{A})\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}}\nabla v(0)\right]\right\rangle, \end{aligned}$$

then we make an error in : $O\left(\frac{1}{k^2}\right)\bar{\nu}(B)^{1/p}$.

- We have :

$$\frac{1}{k}\frac{1}{(2\pi)^2}\int\int_{\{|u|_\infty\geq\beta\sqrt{k}\}}\left|e^{-i\langle\frac{u}{\sqrt{k}},N\rangle}\exp\left(-\frac{1}{2}\langle\Sigma^2u,u\rangle\right)C_k(u)\right|du \leq O(k^{-2})\bar{\nu}(B)^{1/p}.$$

- Hence we have to estimate :

$$\frac{1}{k}\frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2}e^{-i\langle\frac{u}{\sqrt{k}},N\rangle}\exp\left(-\frac{1}{2}\langle\Sigma^2u,u\rangle\right)C_k(u),du.$$

This quantity can be rewritten : $G+H$ with :

$$G := \frac{\bar{\nu}(A)\bar{\nu}(B)}{(2\pi)^2k}\Phi\left(-\frac{N}{\sqrt{k}}\right)$$

and

$$H := \frac{1}{i(2\pi)^2k^{3/2}}\left\langle\nabla\Phi\left(-\frac{N}{\sqrt{k}}\right), \nabla\phi(0)(\hat{P}^r(\mathbf{1}_{\hat{A}}))\hat{\nu}(\hat{B}) + \hat{\nu}(\hat{A})\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}}\nabla v(0)\right]\right\rangle$$

with :

$$\begin{aligned} \Phi(X) &:= \int\int_{\mathbb{R}^2}\exp(i\langle u,X\rangle)\exp(-\frac{1}{2}\langle\Sigma^2u,u\rangle)\,du \\ &= \frac{2\pi}{\sqrt{det(\Sigma^2)}}\exp\left(-\frac{1}{2}\langle(\Sigma^2)^{-1}X,X\rangle\right). \end{aligned}$$

But we have :

$$\nabla\Phi(X) = -\frac{2\pi}{\sqrt{det(\Sigma^2)}}\exp\left(-\frac{1}{2}\langle(\Sigma^2)^{-1}X,X\rangle\right)(\Sigma^2)^{-1}X.$$

Hence we have :

$$\begin{aligned} H = \frac{1}{i2\pi k^{3/2}\sqrt{det(\Sigma^2)}}e^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}N,N\rangle}\Big\langle\frac{(\Sigma^{-1})^2(N)}{\sqrt{k}}, \nabla\phi(0)(\hat{P}^r(\mathbf{1}_{\hat{A}}))\bar{\nu}(B)+ \\ + \hat{\nu}(\hat{A})\mathbb{E}_{\hat{\nu}}\left[\mathbf{1}_{\hat{B}}\nabla v(0)\right]\Big\rangle \end{aligned}$$

Hence, there exists some constant K_1 such that :

$$|H| \leq \frac{K_1}{k^{3/2}}\exp^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}N,N\rangle}\frac{\|N\|}{\sqrt{k}}(\bar{\nu}(B)+\bar{\nu}(A)\bar{\nu}(B)^{1/p}).$$

Again this quantity is null if $N=0$. Finally we get :

$$\begin{aligned} &\left|\bar{\nu}\left(A\cap\bar{T}^{-(k+r)}(B)\cap\{S_{k+r}-S_r=N\}\right) - \frac{\bar{\nu}(A)\bar{\nu}(B)}{\sqrt{det(\Sigma^2)}2\pi k}e^{-\frac{1}{2k}\langle(\Sigma^2)^{-1}N,N\rangle}\right| \leq \\ &\leq K_0\left(\frac{\bar{\nu}(B)+\bar{\nu}(A)\bar{\nu}(B)^{1/p}}{k^{3/2}}\left(\frac{\|N\|}{\sqrt{k}}+\frac{\|N\|^3}{k^{3/2}}\right)e^{-\frac{1}{2k}a\langle N,N\rangle}+\frac{\bar{\nu}(B)^{1/p}}{k^2}\right). \end{aligned}$$

Let us notice thate there exists two real numbers $c>0$ and $c'>0$ such that, for all $x\in\mathbb{R}$, we have : $(|x|+|x|^3)e^{-\frac{a}{2}x^2}\leq c|x|e^{-\frac{a}{4}x^2}$ and $|x|+|x|^3)e^{-\frac{a}{2}x^2}\leq c'e^{-\frac{a}{4}x^2}$.

B Adaptation of Young's construction

In this section, we explain briefly how we can adapt Young's construction. Young's construction for billiard uses estimates of [7]. Because of the complexity of the construction, we only give general ideas without going into details.

1. **Construction of Λ.**

   We take $x^{(0)}$ a well chosen point and two well chosen real numbers $\lambda_1 > 1$ and $\delta > 0$.

   We consider the set Ω of points $z \in \gamma^u(x^{(0)})$ such that $\int_{\gamma^u(x^{(0)})_{|x^{(0)},z}} \cos(\varphi)\, dr \le \delta$. We define :

   $$\Omega_\infty = \{y \in \Omega \ : \ \forall i \ge 0,\ d(\bar{T}^i(y), R_0 \cup \bar{T}^{-1}(R_0)) \ge 2\delta_1 {\lambda_1}^{-i}\}.$$

   We define $\Lambda = \left(\bigcup_{\gamma^s \in \Gamma^s_\Lambda} \gamma^s\right) \cap \left(\bigcup_{\gamma^u \in \Gamma^u_\Lambda} \gamma^u\right)$, where $\Gamma^s_\Lambda := \{\gamma^s_\delta(y),\ y \in \Omega_\infty\}$ with $\gamma^s_\delta(y)$ the set of points $z \in \gamma^s(y)$ such that $\int_{\gamma^s(y)_{|y,z}} \cos(\varphi)\, dr \le \delta$ and where Γ^u_Λ is the set of unstable curves γ^u such that :

   - γ^u intersects each $\gamma^s \in \Gamma^s_\Lambda$;
   - for all $y \in \partial\gamma^u$ and for all $z \in \gamma^u \cap \bigcup_{\gamma^s \in \Gamma^s_\Lambda} \gamma^s$, we have : $\int_{(\gamma^u)_{|y,z}} \cos(\varphi)\, dr \ge \delta$;
   - for all $n \ge 0$, $\bar{T}^{-n}(\gamma)$ is contained in at most three adjacent I_k's with, for all $k \ge k_0$:

     $$I_k := \left\{(i, r, \varphi) \in \bar{M} \ : \ \frac{\pi}{2} - \frac{1}{k^2} < \varphi \le \frac{\pi}{2} - \frac{1}{(k+1)^2}\right\},$$

     $$I_{-k} := \left\{(i, r, \varphi) \in \bar{M} \ : \ -\frac{\pi}{2} + \frac{1}{(k+1)^2} \le \varphi < -\frac{\pi}{2} + \frac{1}{k^2}\right\}$$

     and $I_0 := \left\{(i, r, \varphi) \in \bar{M} \ : \ \frac{\pi}{2}|\varphi| \le \frac{\pi}{2} - \frac{1}{(k_0)^2}\right\}$, k_0 being some fixed integer large enough.

   The set Λ is compact and has positive measure.

2. There exists N_0 such that, for any $m \ge N_0$, Young gives the construction of a return time $R(\cdot)$ in Λ with values in $m\mathbb{Z}^*_+$. Let us notice that her construction is still true for a return time with values in $1 + \mathbb{Z}^*_+ m$.

3. Let $a_0 \ge 1$ such that $\Omega \setminus \bigcup_{k=0}^{a_0} \bar{T}^{-k}(R_0)$ is composed of at least five connected components A such that $A \cap \Lambda$ has a nonnegative length. Let us write $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$ three such components "away" from $\partial\Omega$.

4. Return time for the points in $B_0 := \bigcup_{\gamma^s \in \Gamma^s_\Lambda \ : \ \gamma^s \cap A_0 = \emptyset} (\Lambda \cap \gamma^s)$.

   Adapting the argument used by Young in section 8 of [26] (in particular page 642) and using the ergodicity of the billiard transformation in the plane, we construct $\Lambda_0 \subseteq B_0$ with nonnegative measure and an integer $R_0 \ge \max(a_0, N_0)$ such that, on Λ_0, $R(\cdot) \equiv R_0$ and $S_{R_0} \equiv (0, 0)$.

   Following Young's construction, we construct a return time multiple of R_0 on $B_0 \setminus \Lambda_0$.

5. Return time for the points in $B_1 := \bigcup_{\gamma^s \in \Gamma^s_\Lambda \,:\, \gamma^s \cap A_1 = \emptyset} (\Lambda \cap \gamma^s)$.

   In the same way, using Young's argument and the total ergodicity of the billiard transformation in the plane, we construct $\Lambda_1 \subseteq B_1$ on which the return time is equal to an integer $R_1 \geq 1$ multiple of R_0 and on which we have : $S_{R_1} = (1,0)$. We follow Young's construction with $m = R_1$ for the remaining part $B_1 \setminus \Lambda_1$.

6. Let $B_2 := \bigcup_{\gamma^s \in \Gamma^s_\Lambda \,:\, \gamma^s \cap A_2 = \emptyset} (\Lambda \cap \gamma^s)$.

   We construct $\Lambda_2 \subseteq B_2$ and an integer $R_2 \geq 1$ multiple ot R_1 such that $R(\cdot) = R_2$ and $S_{R_2} = (0,1)$ on Λ_2. We follow Young's construction with $m = R_2$ for the remaining part $B_2 \setminus \Lambda_2$.

7. For the remaining part $\Lambda \setminus (B_0 \cup B_1 \cup B_2)$, we adapt Young's construction to get a return time with values in $1 + R_2 \mathbb{Z}_+^*$.

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